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Serology. — *"A Contribution to the knowledge of the nature of the Tuberculin-reaction."* By Prof. C. H. H. SPRONCK and Miss W. HAMBURGER.

(Communicated at the meeting of December 19, 1925).

A few months ago the American researchers LONG and SEIBERT [1]) recorded that sensitized, normal tissues of tuberculous caviae do not adsorb tuberculin in vitro. They had made experiments with fine-grounded testis-tissue and skin. To 0.8 c.c. of emulsion 0.2 c.c. of tuberculin was added and the mixture was placed in an incubator for 6 hours. Subsequently it was centrifugalized and the supernatant fluid was injected into the testis of a tuberculous cavia and of a healthy one. The healthy animal showed no reaction, but a violent inflammation came forth in the tuberculous animals.

In this way, then, nothing could be observed of tuberculin combination, no more of the generation of a poisonous substance, which according to some was to arise from the tuberculin, when it is combined with its reciprocal antibody and to cause the tuberculin reaction. This induced LONG and SEIBERT to side with the adherents of SELTER's theory, which they consider, however, as "a statement of our ignorance of the true nature of the reaction rather than an explanation of it". Indeed, this somewhat singular theory runs as follows: tuberculin is a specific irritant for the sensitized cells, which poisons the cells without, however, being bound by them.

The allergy for albumins, proteins, micro-organisms, is generally attributed to antibodies, which originate in cells, especially in endothelium cells and can pass into the blood. Allowance should be made for the possibility that a substance that lacks antigenic properties, acquires them by combination with a protein. This has already been demonstrated for lipoid cell-substances.

In the investigation of the tuberculin-sensitivity of the tuberculous organism WASSERMANN and BRUCK's [2]) antigenic-antibody theory clashed with the fact that tuberculin has no antigenic properties or only very weak ones. The degree of allergy which, through repeated injection of tuberculin is difficult to elicit in healthy animals, is but the faint shadow of the sensitivity readily generated by the living bacillus tuberculosis. Nay, the question has been propounded whether the artificial tuberculin-sensitivity is, indeed, a true tuberculin sensitivity. Not without reason, since the

[1]) The Journal of the American med. Association, V. 85, N°. 9.

[2]) Deutsche med. Wochenschrift 1906, Nr. 12.

product called tuberculin, is a mixture of all sorts of substances, and investigators have not yet succeeded in acquiring the pure substance that is the etiological factor of the tuberculin-reaction.

Experiments on analysis of the immunity against tuberculosis [1]), which are still continued, have elucidated this problem. Filtrates of aqueous extracts of tuberculous cavia-tissue appeared to contain a substance that acts in tuberculous caviae like tuberculin, but possesses antigenic properties. In view of its active sensitization it was termed „tuberculan" (SPRONCK). Filtrates of cultures of bacilli tuberculosis contained tuberculin, never tuberculan. The reason assumed was that the bacillus tuberculosis produces in the body a complete sensitizing antigen, in vitro, however, a half, incomplete antigen without antigen power, but attracted and bound by the sensitized cells in the same way as the complete antigen. So in LANDSTEINER's nomenclature, tuberculin would be the hapten of tuberculan. In an earlier paper LANDSTEINER had already suggested the idea that tuberculan might probably be a hapten.

Moreover, we had already ascertained through our own inquiries that under the influence of repeated tuberculin stimuli the sensitized cells of the tuberculous cavia produce an antibody, that is transferred to the blood. This imparts to the serum the property to neutralize the action of tuberculin in vitro and to sensitize healthy caviae passively.

Also RUPPEL and RICKMANN's [2]) tuberculosis-serum possessed the quality to counteract tuberculin in vitro. This serum had been manufactured by sensitizing oxen and mules beforehand through infection with living virulent tubercle bacilli from man, and by obtunding after some interval of time the tuberculin-reaction by an injection of tuberculin, extracts from tubercle bacilli and living bacilli. Such a serum has latterly been applied in MOLLGAARD's sanocrysin-therapy [3]).

When tuberculin is binding with its antibody, the production of a poison in the serum is out of the question. If a poison were produced, the binding would not have been detected, as this can appear only from the harmlessness for the sensitized organism. Another explanation of the symptoms that characterize the tuberculin-reaction had, therefore, to be looked for, and we assumed with DÖRR that the physical consequences of the process of the intracellular binding produces as such the irritation, injury, necrosis of cells, inflammation, fever, and also tuberculin death.

Two years ago it was generally admitted that only living bacilli tuberculosis could sensitize the organism for tuberculin. It has since been proved that it is possible to evoke allergy for tuberculin also by injecting dead bacilli into animals and men. BESSAU [4]) and others hold the view that not the dead bacilli themselves evoke the sensitization, but the morbid growth

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1) Ned. Tijdschr. v. Geneeskunde 1923, II Nr. 10; Medizinische Klinik 1923, Nr. 32.
2) Zeitschr. f. Immunitätsforschung 1910, VI, p. 344.
3) H. MOLLGAARD, Chemotherapy of Tuberculosis. Copenhagen, 1924.
4) Klin. Wochenschr. 19 Febr. 1925.

of tissue, induced by the bacilli; in other words, the tuberculous cells are assumed to produce tuberculan, whereas we think that they form anti-tuberculan. We cannot adhere to this view and will only point to the bacillus tuberculosis of CALMETTE, the so called "bacil bilié", which produces tuberculin and also sensitizes the organism, but by no means induces a morbid growth of tissue. Besides this we believe to have succeeded in obtaining tuberculan from cultures of bacillus tuberculosis also outside the organism.

It is obvious, therefore, that we felt justified in testing the above named results of LONG and SEIBERT. But their method, by which the binding of small quanta of tuberculin could not be demonstrated, has been modified by us in the following way.

We have made experiments with testis-tissue and determined in the first place the smallest quantum of tuberculin (standard tuberculin) that had to be added to an emulsion, consisting of 4 grms of testis-tissue of a healthy cavia and 6 c.c. of a physiological salt solution with which tuberculin could still be demonstrated distinctly in the filtrate (paper) of this emulsion, through intracutaneous injection of 0.1 c.c. into tuberculous caviae. It thereby appeared that the addition of 0.2 c.c. answered our purpose. It should be added that we made use of tuberculous caviae from 700 to 800 grms in weight because older animals will get more susceptible than young ones, and the susceptibility of the older animals persists so to say until death [1]). White animals were preferred, anyhow animals with a white spot on the skin of the belly, destined for intracutaneous injections after previous epilation. After these preliminary experiments we passed on to the definitive ones. A tuberculous cavia was bled to death, 4 grms of testis-tissue was weighed out and crushed in a mortar with addition of 0.2 c.c. of tuberculin, and subsequently 6 c.c. of a physiological salt solution = emulsion T (tuberculosis). In the same way we treated testis-tissue of a healthy cavia of approximately the same weight = emulsion N (normal). Both emulsions were stirred off and on, and were allowed to stand for three hours at room-temperature. Subsequently they were filtrated through paper. With these rather clear filtrates the following experiments were carried out:

1°. An intracutaneous injection was given to a tuberculous cavia, into the upper part of the belly 0.1 c.c. filtrate T, into the lower part 0.1 c.c. filtrate N.

2°. Similar injections were given to a healthy cavia.

3°. Another healthy cavia received under the skin of the upper part of the belly 2 c.c. filtrate T and in the lower part 2 c.c. filtrate N.

As to distinctness the result of these experiments left nothing to be desired. The healthy caviae evinced not the slightest of reaction. In the case of the tuberculous cavia, on the contrary, a distinct tuberculin-reaction (a small cockade with infiltration and a trace of necrosis in the center)

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[1]) R. DEBRÉ et H. BONNET. C.R. de la Société de Biologie, T. 86, p. 485.

manifested itself regularly on the site of inoculation of filtrate N. This reaction persisted for many days. In the upper part of the belly, at the site of inoculation of filtrate T either some red colouring appeared, that faded already after a couple of days, or no reaction followed at all.

It cannot be mistaken, therefore, that in vitro normal, sensitized testis-tissue of tuberculous caviae adsorbs tuberculin. But of the hypothetical poison, that was supposed by some to be produced, we could no more than LONG and SEIBERT detect anything. We have not tried to determine how much tuberculin is adsorbed by sensitized testis-tissue, because this quantity may be considered to depend upon the degree of sensitization. But 1 grm of testis-tissue adsorbed, as we think, certainly not more than 50 mgr. of tuberculin, so that the result of the experiments by LONG and SEIBERT is quite intelligible, as they had added more than 200 mgr. per gram of tissue.

The adsorption of tuberculin by sensitized tissue, which is readily demonstrable by our method, is quite in harmony with the antigen-antibody theory and goes against SELTER's theory. Our simple method, which may still be improved, can also be applied to compare the sensitivity of different organs of a tuberculous animal.

A short time ago DIENES and FREUND [1]) reported inter alia that they could not demonstrate tuberculin in the urine after the injection of an aqueous extract of bacilli tuberculosis into tuberculous caviae apparently on account of binding with sensitized cells ; in healthy caviae on the other hand, which had received an equal dosis of the same extract, they found a large quantum of tuberculin in the urine. Clinical experience also lent support to the antigen-antibody theory. In their recently published "Leerboek der Longtuberculose" VOS and LEUSDEN state : „In practice we can attain to the best and persistent results with the treatment of those patients with whom desensitization can be obtained to high doses of tuberculin" [2]). As regards tuberculin-treatment the tuberculosis-specialists are divided into two groups. The allergists strive for the maintenance of tuberculin sensitivity ; the anergists on the contrary try to obtund it. The allergists receive support from SELTER's theory, the anergists from WASSERMANN and BRUCK's antigen-antibody theory, as modified by us. According to the latter theory the favourable action of the obtusion of the sensitivity, is easy to understand. Allergy of the tuberculous patient indicates a lack of antibodies in the blood, and that, owing to this the cells are not protected against tuberculan, which is transferred spontaneously from the tuberculous foci to the blood. Through an appropriate tuberculin treatment the sensitized cells are incited to secrete antibodies, which circulate in the blood and cause a decrease of the tuberculin-allergy, because the tuberculin that

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[1]) DIENES and FREUND, American Review of Tuberculosis, Baltimore, Bd. 12, Sept. 1925.

[2]) B. H. VOS and J. TH. LEUSDEN, Leerboek der Longtuberculose. Utrecht. A. OOSTHOEK, 1925, Dl. II, p. 233.

is introduced, has already been saturated with antibodies in the blood, and consequently does not reach the cells. The obtunding treatment transfers the process of the binding which is deleterious to the cells, from the cells to the blood, where the binding is not noxious. So the obtusion promotes the rest of the sufferer, because it promotes the relaxation required for the healing of the tuberculous foci, since the foci can no more stimulate and deteriorate each other through spontaneous inoculation of tuberculan into the blood. Thus the generally admitted favourable effect of the rest-cure is accomplished by obtusion of the tuberculin-allergy.

From the Bacteriotherapeutic Laboratory at Zeist.)

Physiology. — *"On the Transitory Arrest of the Heart's Action"*. By Prof. H. ZWAARDEMAKER and W. WITANOWSKI. (Communicated by Prof. H. ZWAARDEMAKER).

(Communicated at the meeting of January 30, 1926).

When the conditions under which the heart functionates are of a sudden considerably altered, it often happens that the organ ceases to beat for some time and resumes its pulsations a few moments later. One receives an impression that the heart wants to accommodate itself before getting used to the new condition.

These transitory conditions have often been studied before, also with the isolated heart. NOYONS and LIBBRECHT [1]) were confronted with them during sudden changes of temperature; during a sudden change of pressure every researcher must have observed them; during a sudden change of osmotic pressure and of concentration of hydrogen-ions ODINOT [2]) noted them; during sudden changes of potassium-concentration in the circulating fluid they are quite common. (NOYONS, LIBBRECHT, BUSQUET) [3]).

Lately we could establish them also with abrupt changes in the concentration of Na and Ca, and also in artificial systems with a change in the concentration of NH_4, Rb, Cs, UO_2, Th., so that it is safe to say that nearly all sudden changes are attended with transitory standstills.

Together with the visible movements the action-currents of the heart disappear in all these cases [4]).

Besides transitory standstills other transition phenomena can appear with abrupt quantitative transitions. (T, p, π, P_H, Conc, Na, Conc. K, Conc. Ca, Conc. Rb, Conc. Cs, Conc. NH_4. Conc. U. Conc. Th.). Among these we reckon, e. g., also the group-formation appearing for some time in the pulsation instead of a transitory arrest. Sudden changes of force or relaxation have also been observed.

We purpose to discuss very briefly the significance of these transition-phenomena.

All of them have one characteristic in common: they are generated by an abrupt quantitative modification in a certain condition and they all have the peculiarity that they appear and disappear critically with extreme rapidity, with a latent period of half a min. We suspect, therefore, that we have to do here with a surface phenomenon, and we

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1) NOYONS and LIBBRECHT, Arch. internat. de Physiol. T. 16, p. 451.

2) Unpublished investigation in the Physiol. Lab. of Utrecht.

3) Only calcium forms an exception. See Mines, Arbeiter, etc.

4) An investigation by STEYNS has lately cleared up the Ca-controversy.

naturally feel inclined to correlate this with the so-called antagonisms, resp. pseudo-antagonisms of the ions. The ions of which this antagonism, resp. pseudo-antagonism has been established are Na, K, Ca. The numerical relations are represented schematically:

A. In the case of single antagonism by the fraction $\frac{Na + K}{Ca} =$ $=$ constant.

B. In the case of threefold antagonism or pseudo-antagonism by the triangle

Geometrically the triangle is the base of a pyramid, whose apex is located in the origin of coordinates of a three-dimensional system of coordinates Na. K, Ca. The base of the pyramid may be displaced in a direction parallel to itself, but a displacement in another direction is not consistent with normal functions.

It should not been imagined, however, that the relations in question $\frac{Na}{Ca} =$ constant, $\frac{K}{Ca} =$ constant, $\frac{Na}{K} =$ constant, must be kept invariable. On the contrary, in practice ample variation is admissible as well in J. LOEB's classical researches as in the most recent ones.

The antagonisms are most often correlated with softening or hardening of the surface layer (H. MEYER, SPIRO, HÖBER), and the pseudo-antagonisms are ascribed to the necessity of maintaining a proper permeability (J. LOEB, H. J. HAMBURGER).

We shall not enter into a discussion of these explanations as we prefer to say something about the phenomena as real facts, premising that we deal only with transition-phenomena that arise with abrupt transitions between the several conditions, in which the antagonism as well as the pseudo-antagonism of the ions, of themselves, are fairly well maintained, so that after some time, say for the space of 60 minutes, regular cardiac action is guaranteed.

A. Inotropies and tonotropies as transition-phenomena conforming to the fractional scheme.

For instance sudden changes of lifting-power are met with both in a positive and in a negative sense. They generally reveal themselves with a staircase. The latent period is short, yet in the cold-blooded heart it is mostly not shorter than some seconds.

A very frequent transition-phenomenon is a sudden tonotropy, sometimes in a positive, sometimes in a negative sense. This may be a staircase or a steep reaction. The latent period is longer, it may last several minutes.

From the occurrence of a staircase and from the rather long latent

period we infer that the surface layer in which this occurs should not be looked for too close to the surface of the cell. Considering that in fraction $\frac{Na + K}{Ca}$ increase of the numerator brings on rapid relaxation, and increase of the denominator induces strengthening of the muscle, we localize this transition-phenomenon in the surface-layer of the fibrils, which is probably also the location of the elastic tonus (LANGELAAN, FEENSTRA).

A long series of other ions may also be included in the numerator of our fraction. In the series $Li : Na : NH_4 : K$ we know the quantitative relation 1 : 3 : 12 : 120 (V. D. BOVENKAMP). Afterwards the series could still be prolonged via Rb to Cs. (SMITS). In the denominator Sr and Ba find a place beside Ca, but Be and Mg appeared to be inactive. Quite foreign ions, viz. U and Th can exert, together with calcium, an influence analogous to that of calcium, not, however, when the calcium is absent.

In the glucose heart of NOYONS [1]) the denominator prevails over the numerator as far as tonus is concerned; as regards force the denominator is inferior to the numerator.

B. Chronotropies as transition-phenomenon conforming to the triangle-scheme:

As already stated, the phenomena themselves admit of a division into three groups:

1°. transitory standstills;

2°. group-formations;

3°. transient increase of the frequency.

The latent period of these phenomena is very short, sometimes even no longer than a few seconds, in a coldblooded heart equal to the periodical time of one pulsation.

The duration is very variable, sometimes the standstill lasts only a few seconds, then again a few minutes up to 20. The group-formation manifests itself after 10 minutes or less, and perhaps persists for half an hour. The increase of the frequency has again a short latent period (1 min.) and continues only for 1, 2 to 5 minutes (J. B. ZWAARDEMAKER).[1])

The phenomena are closely allied, for not only can we let them come and go by the same contrivances, but they can also link up to each other, those with a short latent period, of course, coming first and later on those with a longer period.

Their character is exclusively chronotropic. For instance, instead of Na half the substance can be replaced by Li. The vertex of the triangle is then Li + Na. Partially K can be replaced by NH_4, also by Rb. The vertex is then $NH_4 + K + Rb$. In place of Ca Cs can be used. Entirely new triangles may also appear when we introduce the radio-

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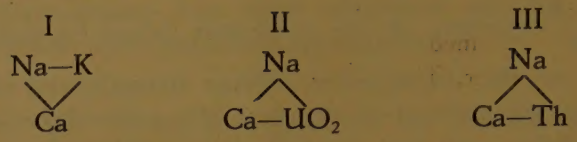
[1]) J. B. ZWAARDEMAKER, Arch. néerl. de physiol. T. 8 p. 414, 1923.

active ions U or Th. Then the following schemata are obtained that are analogous inter se.



Meanwhile this time the transitions are no longer purely quantitative, as the radio-active elements K and U, when acting separately (or all but separately) are, indeed, interchangeable, but when acting simultaneously (which they do of necessity in such transitions), the radio-physiological antagonism asserts itself. The transitions I $\rightleftarrows$ II, resp. I $\rightleftarrows$ III will, therefore, acquire a quantitative and a qualitative character. The transitions I $\rightleftarrows$ III on the other hand are of a qualitative nature.

From what has been set forth here it appears, then, that the following phenomena reveal themselves:

1°. transition-phenomena of a quantitative origin conforming to the fraction scheme;

2°. transition-phenomena of a quantitative origin conforming to the triangle-scheme;

3°. transition-phenomena of a qualitative origin.

I must not omit to call attention to the following remarkable peculiarity:

Wheras the transition-phenomena sub 1 and 2 still show some difference in their mode of appearing (gradual development, resp. recession sub 1 sudden coming and going sub 2), not any outward difference is noticeable in the transition-phenomena sub 2 and 3. We there observe the same suddenness in appearing and disappearing, equal latent period (on an average 30 sec.) equal duration (on an average 1 or 2 minutes), mostly complete relaxation of the muscle during the standstill. Yet they originate in a different way. In the one case the standstill is brought about by the sudden modification in the quantity of a condition, in the other by a sudden modification in the nature of a condition (alpha-radiation, i.e. emission of positively charged, very large ions over against beta-radiation, i.e. negatively charged, very small particles). True, there is no denying the fact that between alpha-, and beta-radiation there is besides a qualitative also a quantitative difference (e.g. in the amount of kinetic energy), but it would not do to attach value to this quantitative difference alone, for in doing so the difference in positive and negative charge would be eliminated, and the biological contrast between the two radiations (radio-physiological antagonism) would not be taken into account. The connection will be readily understood, however, when we examine the frequency of the occurrence of the standstills, i.e. when we compare e.g. a frequency with purely quantitative difference with the frequency of the standstills in case of a complex of quantitative difference and

qualitative difference. The frequency of the standstills amounted, for purely quantitative difference, to 33 % of the expressly examined cases (VAN DISHOECK), over against 50 % when qualitative and quantitative differences were combined. Occasionally the two transitions will appear the one after the other. The artificial Na Ca UO_2-system, and the equally artificial Na Ca Th-system have not yet been dealt with systematically. Provisionally we are impressed with the idea that in the main (not entirely) [1]) they are analogously related to the natural Na K Ca-system.

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[1]) Not entirely, as there is no abrupt increase of frequency in Petromyzon on a sudden removal of the UO_2, resp. Th. from the circulating fluid. In fact also the abrupt transition-standstills and transition-groupings are less frequent in the artificial systems. Meantime there remains a possibility that owing to the slow diffusion of the UO_2- and Th-ions the transitions are not comparable.

Chemistry. — *"The Potentiometric Determination of the Hydrogen Ion Concentration at Higher Temperatures"*. By I. M. KOLTHOFF and F. TEKELENBURG. (Communicated by Prof. H. R. KRUYT).

(Communicated at the meeting of January 30, 1926).

1. In an investigation on the variation of the constant of dissociation of organic acids and the p_H of buffer mixtures at higher temperatures we have performed measurements of the hydrogen ion concentration, both with the use of the hydrogen and of the quinhydron electrode.

The E. M. F. of one of these electrodes is measured by some standard electrode or other, and then the p_H is calculated according to the formula:

$$p_H = \frac{\pi - \pi_0}{\frac{RT}{F}} = \frac{\pi - \pi_0}{0.0001983\,T}.$$

π is the E. M. F. measured, compared with the standard electrode.

π_0 is the E.M.F. of the standard electrode compared with the electrode which is normal as regards hydrogen ions, T representing the absolute temperature.

The formula holds for different temperatures. If from the measured value of π it is desired to calculate the p_H, we must know the value of π_0 at the temperature T. The variation of π with the temperature depends on the temperature coefficient of the electrode which is normal as regards hydrogen ions, and on that of the standard electrode; the temperature coefficient of π_0 being equal to the algebraic sum of the two electrodes separately.

The value of π_0 at room temperature is not known with great certainty. Internationally with use of 0.1 N calomel electrode the value found by SÖRENSEN [1]) is universally accepted as standard electrode at 18°. This value has been calculated from the E. M. F. of the circuit: H_2 electrode — 0.01 N HCl + 0.09 N KCl/— 0.1 N calomel electrode. Besides SÖRENSEN has calculated the p_H of the hydrochloric mixture from the electric conductivity according to ARRHENIUS' isohydric principle, finding a value of 2.038. We should, however, bear in mind that the electric conductivity does not convey a correct impression of the activity of the hydrogen ions.

In this connection we may especially refer to an important communication by G. N. LEWIS, T. B. BRIGHTON, and R. L. SEBASTIAN [2]), who

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[1]) S. P. L. SÖRENSEN, Compt. rend. Lab. de Carlsberg 8, bl. 22 (1909); and also SÖRENSEN and K. LINDERSTRÖM-LANG, Ibid. 15, No. 6 (1924).

[2]) LEWIS, BRIGHTON and SEBASTIAN, J. Amer. Chem. Soc. 39, 2245 (1917).

have carried out measurements with diluted hydrochloric acid solutions without chloride of potassium, and have calculated from it a value of π_0, 2 m. Volt lower than that according to SÖRENSEN.

We hope to return more at length to the correct standard value later on. For the present we have assumed the value π_0 given by SÖRENSEN, to be correct.

For the results which have been mentioned in these communications it is, for the rest, immaterial which standard value is assumed at 18°, as it was chiefly the intention to get to know the temperature coefficient of π. The only investigation on the temperature coefficient of the N-hydrogen electrode has been performed by KOEFOED.

His results have been communicated by SÖRENSEN [1]). The measurements have been made against 0.1 N calomel electrode in an air-thermostat. Further particulars have not been mentioned. From the known temperature coefficient of the N-calomel electrode SÖRENSEN calculates that the potential of the normal hydrogon electrode at increasing temperature diminishes 0.85 m. Volt per degree. (SÖRENSEN has made here a mistake about the sign, for the potential of the N-hydrogen electrode increases with the temperature).

E. BILLMANN and J. KRARUP [2]) have measured the temperature coefficient of the element hydrogenelectrode-quinhydron electrode. They found that the E. M. F. of this element decreases by 0.74 m.Volt per degree at rising temperature.

As there are so few data in the literature on the temperature coefficient of the hydrogen- and quinhydronelectrode, we have made a thorough investigation of this question.

In the course of this examination we have determined the coefficients of these two electrodes and of the *n*-calomelelectrode both separately, as of elements built up of different electrodes. In this way it was easy to check the reliability of the results. It may already be mentioned here that we found a good agreement with the values which have been communicated by SÖRENSEN, BILLMAN and their collaborators.

2. *Apparatus.* In the application of the compensation method we always used a student's potentiometer of LEEDS & NORTHRUP Company, with accessory galvanometer. The readings are very sensitive generally up to 0.05 m. Volt. In all cases the measurements could be made with an accuracy of 0.1 m.Volt. The scalar division of the potentiometer was checked by comparing the readings with those obtained by 2 precision rheostats instead of using the potentiometer.

As electrodes we used the N-calomel-, the hydrogen-, and the quinhydron electrode. All the determinations were carried out in a water thermostat, in which a layer of paraffin oil prevented too strong evapo-

[1]) SÖRENSEN, Compt. rend. Lab. de Carlsberg, No. 9 (1912); cf. SÖRENSEN and LINDERSTRÖM-LANG l. c.

[2]) E. BILLMANN and J. KRARUP, J. Chem. Soc. 125, 1954 (1924).

risation of the water. As we gradually raised the temperature in every series of experiments, an air thermostat was not suitable for our purpose, because the time that elapses before the different vessels have assumed a constant temperature, is much too long.

In all the cases the temperature was constant up to 0.1° C., a more accurate regulation of the temperature was not necessary for our purpose.

As the forms of the electrodevessels generally used are not very suitable when a water thermostat is worked with, we have used somewhat different models, so that electrode and liquid were entirely in the thermostat. The electrode vessels were made of Jena normal glass, and were cleaned by steam before the filling. The tube that served for the calomel electrode had an inner diameter of 2 cm. and was 7 cm. long. It was closed by a rubber stopper with 2 perforations; through one passed the capillary, in which at the bottom a piece of platinum had been fused which formed contact with the mercury; through the other the connecting siphon could be adjusted, which joined the normal calomel electrode with the other electrode. The filling of the "half-cell" took place with the usual precautions. When the quinhydron electrode was used, we took an analogous vessel as has been described for the calomel electrode. The best results appeared to be obtained when a piece of platinum gauze was used. The quinhydron had been prepared in the usual way out of hydroquinone and ferri-ammonium sulphate, and then recrystallized out of water. The vessel in which the measurements with the hydrogen electrode were performed, was wider than those described above. The vessel was also closed with a rubber cork, in which several perforations had been made. Through one passed the inlet tube for the hydrogen. This tube was bent to a U-shape at the bottom, and care was taken that a part of the platinized platinum filament electrode extended into the tube. In this way the electrode was continually in contact with the in-flowing hydrogen. In order to prevent evaporation of the liquid in the electrode vessel at higher temperature, we always led the hydrogen through a water bottle placed in the thermostat, and filled with the same liquid as the electrode vessel. Through another perforation of the cork passed a narrow tube, through which the hydrogen could escape, and through a third perforation the connecting siphon with the standard electrode could be passed.

The measurements with the hydrogen electrode were easily reproducible to a 0.1 m. Volt.

The siphons, which served to connect the electrodes with each other, were U-shaped tubes, with a bore of 1 to 2 mm. The tubes were filled with solutions containing 1.75 resp. 3.5 N. chloride of potassium and 2 % agar. The agar was dissolved in the heated salt-solution, then after filtration the heated liquid was sucked up into the tubes. At room-temperature the contents get solid. Also in the measurements at 60° to 70°

these tubes could be used. By always using two siphons, one containing 1.75 N. and the other 3.5 N. chloride of potassium, the potential of diffusion according to N. BJERRUM [1]) could be eliminated.

In the measurement of the temperature coefficient of an element, part of the siphon was in the air, so that the contents had not the same temperature everywhere. As it was possible that in this way errors might arise, we have also carried out measurements, in which this drawback was obviated. The apparatus used is represented in Fig. 1.

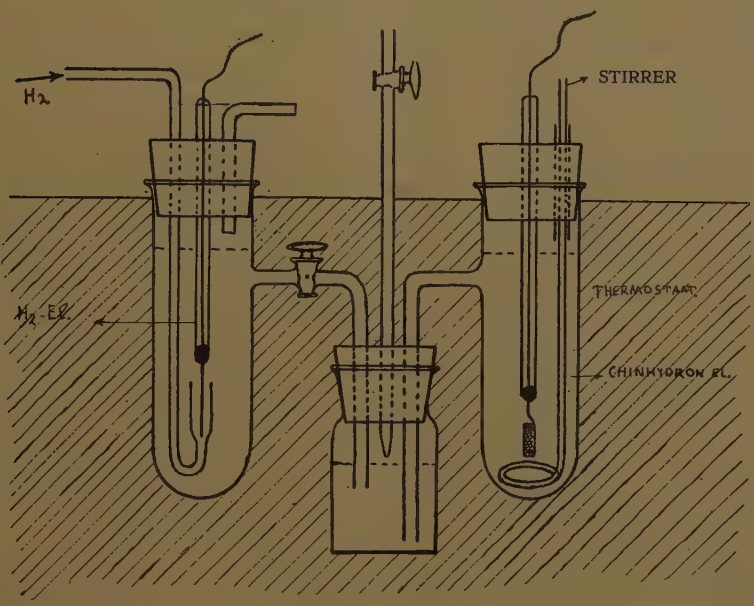

Fig. 1.

One leg served as receptacle for the hydrogen, the other as receptacle for the quinhydronelectrode. The two vessels were filled with the same liquid. To prevent diffusion of quinhydron into the hydrogen electrode, a connecting vessel was applied, also filled with the same liquid as that with which the electrodes were in contact. Hence a diffusion potential could not appear. In the vessel in which the quinhydronelectrode was immersed, a glass stirrer was placed, so that the liquid could be stirred at the higher temperatures, and could be quickly saturated with quinhydron in this way. The connecting tap, which was also immersed in the thermostat, was opened during the measurement by the aid of a wooden fork, and closed again after the reading. The whole element could be immersed in the thermostat. It now appeared that we practically obtained the same results with this element as with our usual arrangement.

In the measurement of the temperature coefficient of one electrode

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N. BJERRUM, Z. physik. Chem. 53, 428 (1905).

separately, the electrode of comparison was placed in a DEWAR vessel in melting ice, so that the temperature remained constant during the series of measurements.

3. *The Reproducibility of the Electrodes at Higher Temperatures.*

In measurements of ionconcentrations as a rule some calomel electrode or other is used as standard electrode.

In the measurements of ion concentrations at varying temperatures our experiences with these electrodes have not been favourable. When the normal calomelelectrode is heated, without moving or shaking, it takes a pretty long time, before the state of equilibrium is reached. Sometimes we had to wait as much as two hours or more until the potential was constant. Of course the establishment of the equilibrium could be accelerated, by a few minutes' shaking in the thermostat at the higher temperatures. It appeared, however, that after the shaking, the electrode had a too noble potential due to the suspended calomel. To obtain reproducible values we had to wait till all the calomel had settled down the bottom, which may take a pretty long time. If the measurement is made while the liquid is still turbid of the suspended calomel, errors may ensue of 1 to 1.5 m.Volt. A more serious drawback is, that the calomel electrode, after having been at high temperature, no longer shows the original potential after having been cooled down to room temperature. Thus we have repeatedly found that an electrode that had been heated to 60° had a potential at 18° which was 5 m. Volt more positive than originally.

This deviation is a consequence:

1. of the super-saturation of the liquid with calomel after cooling;
2. of the changed equilibrium:

$$Hg^{++} + Hg \rightleftarrows Hg_2^{++}.$$

At higher temperature the state of equilibrium changes and it takes a long time after the cooling, before the original state has again been reached. When the liquid is shaken for a long time after the cooling and when it is allowed to settle, or when the electrode liquid is refreshed, the original values are found back.

It appears from the above:

1. *that when a calomel electrode is used as standard, the vessel must always be kept at the temperature at which the measurement is made.* Only in this way one can be sure that the potential remains constant.
2. that when varying temperatures are worked with, the use of the calomel electrode is attended with many difficulties. For series measurements at different temperatures it is therefore *not* to be recommended as a standard.

For these reasons we have also worked with another standard electrode; we took a quinhydron electrode in contact with a liquid containing

0.01 N. hydrochloric acid and 0.09 N. potassium chloride. Such an electrode has already been recommended by S. VEIBEL [1]) as standard, and he has also already determined its temperaturecoefficient between 0° and 18° in connection to the hydrogen electrode.

With this standard-quinhydron electrode we have obtained very satisfactory results. Up to a temperature of 40° the potential remains constant for a long time; at higher temperatures, however, changes may be observed when the electrode is left for some time; the electrode becomes less noble. This may be partly a consequence of an evaporation of quinone, but besides a reaction seems to take place between quinone and hydrochloric acid. For after the heating of quinhydron with the electrode liquid at 60° in a closed vessel it is found that the potential has considerably fallen after cooling.

After the slow heating of a quinhydron electrode to 50° we have repeatedly observed that after cooling to 18° the potential had fallen from 6 to 8 m. Volt. When the quinhydron electrode is used at higher temperatures, it is, however, very well possible to obtain reproducible values, when the electrode is refreshed resp. at 50°, 60°, and 70°. Every time fresh quinhydron is added, the electrode is shaken for about 3 minutes, after which the potential remains constant for a short time.

As the state of equilibrium is reached so quickly, and the manipulation of the quinhydron electrode is very simple, we can recommend it as standard electrode. In series of measurements which were made on different days under the same conditions, we found results agreeing to at least 0.2 m. Volt.

We may still mention that instead of the hydrochloric acid potassium chloride mixture we have also tried several buffer solutions as electrode liquids. Here, too, we found that at higher temperatures the potential remained constant only for a short time.

In conclusion we may still remark that in principle it is most advisable to use a hydrogen electrode at higher temperatures as standard. L. MICHAELIS [2]) has already recommended an acetic acid-acetate mixture for this purpose; we, however, prefer SÖRENSEN's hydrochloric acid-potassium chloride mixture (0.01 NHCl + 0.09 NKCl). The presence of the potassium chloride causes the diffusion potential to be exceedingly small.

4. *Results.*

In the first place we have measured the temperature coefficients of the normal calomel electrode with due observation of the different precautions described under 3. For this purpose we have worked with 2

1) S. VEIBEL, J. Chem. Soc. 123, 2203 (1923).

2) L. MICHAELIS, Die Wasserstoffionen-konzentration, 1e Aufl. Berlin 1914, p. 148.

different electrodes, a third electrode being kept constant at 0°. It was first ascertained that the electrodes at 18° connected with each other had an E.M.F. of 0.0 m.Volt. At higher temperatures the potential of the calomel electrode becomes more positive. Some results are laid down in the following tables:

Temperature coefficient N-calomel electrode.

N. C. E. I t	N. C. E. II t	E. M. F. in Volts	Temp. coeff.
0°	18.6	0.0114	0.00061
0	26.2	0.0155	0.00060
0	30.3	0.0175	0.00057
0	40.2	0.0231	0.00057
0	50.2	0.0288	0.00057
0	60.2	0.0343	0.00056

mean: 0.00058
(between 0° and 30°: 0.00059)

N. C. E. III	N. C. E. IV	E. M. F.	Temp. coeff.
0°	26.2	0.0161	0.00061
0	30.4	0.0184	0.00060
0	40.2	0.0229	0.00057
0	50.1	0.0291	0.00058

mean: 0.00059
(between 0° and 30°: 0.00060)

As a mean we find for the normal-calomel electrode a temperature coefficient of 0.00059, which value is in good harmony with the result of the accurate measurements of TH. W. RICHARDS [1]), who found a coefficient of 0.00061.

In the next table we state the results obtained in the measurement of the temperature coefficient of the quinhydron electrode in the standard hydrochloric acid mixture. The measurement was made by comparison with a quinhydron electrode, which was kept at 0°. Originally the two electrodes had, at 18°, an E.M.F. of 0.0 m.Volt as compared with each other.

[1]) TH. W. RICHARDS, Z. physik. Chem. 24, 49 (1897).

Temperature coefficient quinhydron electrode.

Quinhydron electrode I t	Quinhydron electrode II t	E. M. F. in Volts	Temp. coeff.
0°	18.0°	—0.0054	—0.00030
0	25.0	—0.0074	—0.00030
0	30.0	—0.0092	—0.00031
0	40.0	—0.0123	—0.00031
0	50.0	—0.0152	—0.00030
0	60.0	—0.0183	—0.00030
			average —0.00030

In another series — 0.00031 was found as a mean. We have also carried out a series of measurements against the N-calomel electrode, which latter was kept at 0°. The mean value, which then was found, was — 0.00033. From all the observations which were then made the conclusion may be drawn that the quinhydron electrode has a temperature coefficient of 0.00031.

We then determined the temperature coefficient of the hydrogen electrode, which was in contact with the standard hydrochloric acid mixture. The measurements were carried out against the quinhydron electrode in the same mixture kept at 0°.

In the third column the directly measured E.M.F. is recorded, in the fourth column the recalculated values for a pressure of the hydrogen of 1 atmosphere (hence corrected for barometer position and vapour pressure of the water).

Temperaturecoefficient of the hydrogen electrode.

Quinhydron-electrode t	Hydrogen-electrode t	E. M. F. in volts	E. M. F. in volts corrected B = 756.8	Temp. coeff.
0°	18°	0.7093	0.7096	
0	25	0.7060	0.7064	+ 0.00046
0	30	0.7035	0.7041	+ 0.00046
0	40	0.6985	0.6995	+ 0.00046
0	50	0.6930	0.6948	+ 0.00047
0	60	0.6865	0.6896	+ 0.00052
0	70	0.6793	0.6847	+ 0.00049
				+ 0.00047

In 3 other series of measurements, the particulars of which will be communicated in the thesis for the doctorate of one of us (T.), the mean values of 0.00047, resp. 0.00046, resp. 0.00045 were found. As total average between 0° and 70° we can therefore assume a value of 0.00046 ± 0.00001.

As the quinhydron electrode in the hydrochloric acid-potassium chloride mixture has a temperature coefficient of — 0.00031, the hydrogen electrode of + 0.00046, the element chinhydron-hydrogen electrode both in the same liquid must have a temperature coefficient of — 0.00077 ± 0.00002. We now carried out a great number of measurements between 18° and 70°; in this it was necessary to bring the quinhydron electrode every time in contact with fresh liquid at 50°, 60°, and 70°. If this was not done, irregularities were found.

In TEKELENBURG's thesis for the doctorate all the values will be given: we will mention here only the mean values found in the different series of observations: —0.00074; —0,00076; —0,00077; —0.00077; 0.00075. As total mean we find therefore a temperature coefficient for the element quinhydron-normal-hydrogen-electrode of — 0.00076 ± 0.00001 Volt. Hence this value is in accordance with that which we had calculated from the temperature coefficients of the two electrodes separately. E. BILLMANN and KRARUP (loc. cit.) found, between 0° and 40°, a temperature coefficient of the same element of — 0.00074, which is in satisfactory agreement with the value found by us. In order to ascertain whether there is an error in the determination due to the connecting siphon being only partially immersed in the thermostat, we also made a number of series of observations with the element as represented in Fig. 1. We then found, in perfect harmony with the value given above, a mean temperature coefficient of — 0.00076.

Besides, we also made series of measurements, in which the hydrogen and quinhydron electrode were in contact with buffer mixtures instead of with the standard hydrochloric acid mixture.

In this way the following values were found:

liquid in 0.1 molar secondary sodium citrate $p_H = 4.96$;
temperature coefficient element — 0.00076
liquid in 0.1 molar sodium bi-tartrate
temperature coefficient element — 0.00075.

These values are, therefore, in good agreement with those that were found with the standard hydrochloric acid mixture.

We will still state here that the temperature coefficient of the hydrogen electrode in the standard hydrochloric acid mixture is not equal to that of the *normal* hydrogen electrode.

For if the p_H of the standard mixture is 2.031, and remains unchanged

at higher temperatures, the E. M. F. is with regard to the normal hydrogen electrode

$$\pi = -2.038 \times \frac{RT}{F}.$$

From this formula and the temperature coefficient of the hydrogen electrode in the standard mixture, we then find a coefficient for the *normal hydrogen electrode* of $+(0.00047 + 0.0004) = +0.00087$, while SÖRENSEN has derived a coefficient of the temperature of 0.00085 from KOEFOED's observations (loc. cit.). Consequently the agreement is excellent.

Summary.

1. At higher temperatures the use of the calomel electrode as standard electrode is not to be recommended. The state of equilibrium is only slowly reached with change of temperature, calomel having influence on the potential in suspended condition. In particular the equilibrium is established very slowly when an electrode of a higher temperature is brought to a lower.

2. In series measurements the quinhydron electrode in a liquid containing 0.01 N of hydrochloric acid 0,09 N of potassium chloride (standard hydrochloride mixture) is to be recommended as comparison electrode. At a temperature higher than 40° C. the potential however, changes comparatively quickly with the time, so that at higher temperatures the liquid must be repeatedly refreshed. In principle the hydrogen electrode is, therefore, to be preferred in the standard acid mixture.

3. The temperature coefficient of the N-calomel electrode was found equal to $+0.00059$ (RICHARDS $+0.00061$); of the quinhydron electrode in the standard acid mixture to -0.00031; of the hydrogen electrode in the same mixture to $+0.00046$. From the last value we calculate a temperature coefficient of the normal hydrogen electrode of $+0.00087$ (SÖRENSEN and KOEFOED 0.00085).

4. Of the element quinhydron electrode — hydrogen electrode a temperature coefficient of -0.00076 was found. (BILLMANN and KRARUP -0.00074).

Pharmaceutical Laboratory of the University.

Utrecht, January 1926.

Mathematics. — *"A Representation of the Congruence of the Twisted Cubics which Pass through Four given Points and which Cut a given Straight Line Twice"*. By Dr. G. SCHAAKE. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of January 30, 1926).

§ 1. The twisted cubics k^3 which pass through four given points $H_1, \ldots, H_4$ and which cut a given straight line l twice [1]), may be represented on the points of a plane α in the following way. We assume a one-one correspondence between the points P of l and the points Q of a conic k^2 in α. To a curve k^3 which cuts l in the points P_1 and P_2, we associate the point of intersection K of the lines q_1 and q_2 which touch k^2 at the points Q_1 and Q_2 corresponding resp. to P_1 and P_2.

§ 2. If P_1 lies in the point of intersection S_1 of l with the plane $\beta_1 \equiv H_2H_3H_4$, the curve k^3, which in this case has four points in common with β_1, must degenerate into a conic $k_1{}^2$ of the pencil of which H_2, H_3, H_4, and S_1 are the base points and a straight line k_1 which cuts $k_1{}^2$ outside l, passes through H_1, and cuts l in P_2. Any conic of the said pencil is a part of one degenerate individual of our system. In this way we find ∞^1 degenerate curves k^3, which are represented on the points of the tangent t_1 to k^2 at the point T_1 which corresponds to S_1.

Accordingly the curves of our system which are degenerate in a conic and a straight line, form four systems which are represented on the four tangents $t_1, \ldots, t_4$ to k^2 at the points $T_1, \ldots, T_4$ that are associated to the points of intersection $S_1, \ldots, S_4$ of l with the four sides $\beta_1, \ldots, \beta_4$ of the tetrahedron of which $H_1, \ldots, H_4$ are the angular points.

The conic $k_1{}^2$ may degenerate into S_1H_2 and H_3H_4, and is then completed to a curve k^3 by the straight line H_1S_2 in β_2. Together with H_3H_4 this line forms a degenerate conic $k_2{}^2$ of the pencil which has H_3, H_4, H_1, and S_2 as base points. Consequently we have here a k^3 degenerate in three straight lines, which is represented on the intersection T_{12} of t_1 and t_2.

Hence our system contains six curves k^3 which are degenerate in three straight lines. They are represented on the six points of intersection T_{ik} of the lines $t_1, \ldots, t_4$. The curve k^3 corresponding to the point T_{ik} consists of the edge opposite to H_iH_k and of the lines S_iH_k and S_kH_i.

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[1]) The congruence which is formed by these curves, has been treated by Prof. JAN DE VRIES: A Congruence (1,0) of Twisted Cubics, these Proceedings 26, 126.

§ 3. The tangent at a point Q of k^2 is the image of the system of the ∞^1 curves k^3 which pass through the point P of l corresponding to Q. These curves form the quadratic cone $\varkappa$ which has P as vertex, which passes through the points $H_1, \ldots, H_4$, and which contains the straight line l, as it must have seven points in common with $\varkappa$, to wit two points which coincide in P, one point of intersection with l outside P, and the four points $H_1, \ldots, H_4$.

The system Σ_1 of the ∞^1 curves k^3 which cut a given line m, is represented on a curve of α. As through a given point P of l there pass two curves of Σ_1, which cut m in the two points of intersection of m with $\varkappa$, the aforesaid image curve has two points in common with any tangent to k^2.

Accordingly the system Σ_1 of the ∞^4 curves k^3 which cut a given straight line m, is represented on a conic m^2 of α.

To a curve of Σ_1 which cuts m twice, there would correspond a node of the image curve of Σ_1. As a rule a conic m^2 is not degenerate, as in this case our congruence would be compound, and consequently it does not generally contain any double point.

There is, therefore, no twisted cubic which passes through four given points and which has two given straight lines as chords [1]).

As the conics m^2 corresponding to two different lines m have four points in common, we find:

There are four twisted cubics which pass through four given points, have a given chord, and cut two given straight lines.

The points of k^2 are the images of the curves k^3 which touch l. Now a conic m^2 has four points in common with k^2.

There are accordingly four twisted cubics which pass through four given points, touch a given straight line, and cut another given line.

§ 4. The points K of a straight line k of α are the image points of ∞^1 curves k^3, which form a system S. As the pairs of points (Q_1, Q_2) which correspond to the points K of k, define an involution with carrier k^2, the pairs of points of intersection of l with the curves of S form an involution on l. The locus of the curves of S is a quadratic surface σ^2, as k has two points in common with a conic m^2 and there are, therefore, two curves of S which cut a given straight line m. This surface σ passes through the four points $H_1, \ldots H_4$ and it contains the line l. Each of the ∞^2 quadratic surfaces which satisfy the latter conditions, contains a system of ∞^1 twisted cubics k^3 which pass through $H_1, \ldots, H_4$ and have l as chord, which system is always represented on a straight line of α.

A straight line k of α is the image of a system S of twisted cubics of a quadratic surface σ which passes through $H_1, \ldots, H_4$ and contains l.

[1]) For these results and other derived numbers cf. SCHUBERT: Kalkül der Abzählenden Geometrie, pp. 176 and 182.

The individuals of S are those twisted cubics of σ which pass through $H_1, \ldots, H_4$ and have l as chord.

The surfaces σ which correspond to the lines k of α, form a net.

The curves of S have one point in common with any straight line of σ which cuts l. Hence a line k of α forms a degenerate conic m^2 with any tangent q to k^2. For the system of curves k^3 cutting the straight line of σ which passes through the point P of l that corresponds to q and which is different from l, consists of the system S associated to k and of the curves k^3 through P.

To the straight lines of a plane pencil of α there correspond the surfaces σ of the pencil the base curve of which consists of l and the curve k^3 which is associated to the vertex of the pencil. The surfaces σ corresponding to the two tangents of k^2 in the plane pencil, are the quadratic cones which project k^3 out its two points of intersection with l.

A surface σ is the locus of the curves k^3 which cut a straight line through a point H_i outside H_i. This line is the straight line of this surface belonging to the same scroll as l which passes through H_i and it may coincide with any straight line through H_i. The two points of intersection of a straight line k with k^2 or with a conic m^2 are accordingly the images of two curves of the system associated to k^2 or to m^2 which cut a straight line through H_i outside H_i.

Accordingly the biquadratic surfaces of the twisted cubics which pass through four given points and which, besides, have a given chord and cut a given straight line or touch a given straight line, have a double point in each of the four given points. These surfaces have b resp. as nodal line and as cuspidal line.

If the line m is a chord of a curve k^3, m and l belong to the same scroll of a surface σ, the individual of the pencil containing k^3 which passes through a point of m outside k^3. In this case m is cut twice by any of the ∞^1 curves k^3 of σ, so that the conic m^2 corresponding to m is degenerate in a double line k. The chords of the curves k^3, i. e. the straight lines of the surfaces σ which belong to the same scroll as l, form a quadratic complex C^2. For there is a pencil of surfaces σ which pass through a given point of space. The base curve of this pencil consists of l and the curve k^3 which passes through the chosen point, and the lines of the surfaces σ through this point which belong to the same system of generatrices as l, form the quadratic cone which projects the said curve k^3 out of the point in question. Evidently the complex C^2 has cardinal points H_i and it contains the line l. The complex cone of which the vertex lies in a point of l, contains l and the four points H_i. C^2 is the tetrahedral complex with tetrahedron of singularities $H_1H_2\,H_3H_4$ to which l belongs [1]).

As a straight line k of α has two points in common with k^2, a surface

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[1]) Cf. STURM: Liniengeometrie, 1 p. 335.

σ contains two curves k^3 which touch l, hence also two which touch a generatrix of the system which contains l. C^2 is particularly the complex of the tangents to the curves of our congruence. Any straight line of C^2 is a tangent to two curves k^3.

§ 5. The curves of our congruence which touch a given plane β and which, accordingly, form a surface, are represented on the points of a curve of α. The number of points of intersection of this curve with a line k of α is the number of curves k^3 touching β which lie on the surface σ corresponding to k. Now the curves k^3 of σ cut the plane β in the point-triplets of a cubic involution, which has the intersection of σ and β as carrier. As this involution contains three triplets with two coinciding points, there lie on σ four curves k^3 touching β, so that the image curve of the system Σ_2 of the curves k^3 which touch β, is a curve b^4 of the fourth order.

The image point of a curve k^3 which touches β, always lies on a generatrix of a plane pencil chosen at random in α. Consequently any curve of Σ_2 lies on one of an arbitrary pencil of surfaces σ.

This pencil intersects β along a pencil of conics of which one of the base points lies in the point $l\beta$. The curves of Σ_2 touch β in the points of a curve which has four points outside the base points in common with any conic of the latter pencil. The point $l\beta$ is a double point of the locus of the points of contact. For on the cone $\varkappa$ which has $l\beta$ as vertex and which contains l and $H_1, \ldots, H_4$, there lie two curves k^3 each of which touches one of the common straight lines of $\varkappa$ and β. We see that the locus of the points where the curves of Σ_2 touch the plane β, is a cubic with a double point in $l\beta$.

Consequently the system Σ_2 and the curve b^4 are rational. Any surface σ through a curve k^3 which is associated to a double point of b^4, to which surface there corresponds a straight line of α through this double point, has the property that two of the four curves of Σ_2 which lie on an arbitrary surface σ, coincide in the curve associated to the double point. Hence two of the double points of the involutory $(2, 2)$-correspondence on the intersection of σ with β, wich we get by associating to each other the points of intersection of the same k^3 with β, coincide in the point of contact of the k^3 which corresponds to the double point of b^4. This k^3 has, therefore, three points of intersection with β coinciding in its point of contact, so that it osculates β.

Now the rational curve b^4 has three double points.

There are accordingly three twisted cubics which pass through four given points, have a given chord and osculate a given plane.

To the tangent to b^4 at an arbitrary point of this curve there corresponds the surface σ through the curve k^3 associated to this point which touches β in the point of contact of k^3. For the intersection of this surface σ and β consists of the lines which join the point of contact

with $l\beta$ and with the point of intersection of k^3 and β. The latter straight line is touched by two curves k^3 of σ. The other two curves k^3 of σ which touch β, coincide in the curve which is associated to the point chosen on b^4.

As a pencil of surfaces σ contains three individuals which touch a plane β, in a plane pencil of α there lie three tangents of b^4, so that b^4 is of the third class. Hence the three double points of b^4 must be cusps. This may also be seen by observing that through a k^3 which corresponds to a double point of b^4, there passes only one surface σ touching β. This surface cuts β along the tangent in β of k^3 and the join of the point of contact and $l\beta$. One of the two curves k^3 of σ which touch the former straight line, coincides with the k^3 which corresponds to the double point of b^4, so that three of the curves k^3 of σ which touch β, coincide with this curve and this surface σ is accordingly associated to the tangent of b^4 in the double point in question.

The cone $\varkappa$ containing l and $H_1, \ldots, H_4$ which has $l\beta$ as vertex, is represented on a tangent to k^2 which is a double tangent of b^4. For $\varkappa$ contains only two curves k^3 of Σ_2, those curves k^3 of $\varkappa$ which touch the two common lines of $\varkappa$ and β in the vertex of $\varkappa$.

The system Σ_2 of the curves k^3 of our congruence which touch a given plane β, is represented on a biquadratic curve with three cusps. The double tangent of this curve is at the same time the tangent to k^2 at the point associated to $l\beta$.

If the points of contact of the double tangent of b^4 are real, b^4 has only one real cusp; if the said points of contact are imaginary b^4 has three real cusps.

Hence:

The three curves of our congruence that osculate a given plane β, are real if the cone $\varkappa$ which contains l and $H_1, \ldots, H_4$ and which has $l\beta$ as vertex, has no real straight line in common with β. If $\varkappa$ cuts the plane β along two real lines, two of the three curves of our congruence which osculate β, are imaginary.

If β touches the cone $\varkappa$, the image curve of Σ_2 degenerates into the tangent of k^2 which represents the system of curves k^3 of $\varkappa$, all of which touch β, and into a cubic with one cusp of which the said tangent of k^3 is an inflectional tangent. The point of inflection is the image of the curve k^3 on $\varkappa$ which touches the line common to $\varkappa$ and β in the vertex of $\varkappa$ and which, accordingly, osculates β there. Two of the three curves k^3 which osculate an arbitrary plane, coincide in this curve; the third is associated to the cusp of the said cubic.

The three cuspidal tangents of b^4 pass through one point. The three surfaces σ which pass through the curves k^3 that osculate β and which touch β, belong, therefore, to a pencil and contain the same curve k^3.

As b^4 has eight points in common with a conic m^2 and with k^2 we find:

There are eight twisted cubics which pass through four given points, have a given chord, cut a given straight line, and touch a given plane.

There are eight twisted cubics which pass through four given points and touch a given straight line and a given plane.

Two curves b^4 have sixteen points of intersection.

Accordingly there are sixteen twisted cubics which pass through four given points, have a given chord, and touch two given planes.

§ 6. The chords of the curves k^3 of a surface σ through a point P form the quadratic complex-cone of the tetrahedral complex C^2 with vertex in P. For these chords belong certainly to C^2 and, inversely, the curve k^3 through a point of intersection of a generatrix of the said complex-cone with σ which cuts this generatrix twice, must lie entirely on σ. This k^3 passes, therefore, through the second point of intersection of the above mentioned generatrix and σ.

The complex-cone of C^2 and the cone of the tangents to σ, both of which have their vertex in P, cut each other along four straight lines. These are tangents through P of curves k^3 of σ. As, accordingly, on a surface σ there lie four curves k^3 of which a tangent passes through an arbitrary point P, the image curve of the system Σ_3 of the ∞^1 curves k^3 of which the developable surfaces pass through a given point P, has four points in common with any straight line of α. Consequently the system Σ_3 is represented on a biquadratic curve p^4 of α.

The numbers of points of intersection of a curve p^4 with a conic m^2, k^2, a curve b^4 and another curve p^4 lead resp. to:

There are eight twisted cubics which pass through four given points, have a given chord, have a tangent through a given point, and cut a given straight line.

There are eight twisted cubics which pass through four given points, touch a given straight line, and have a tangent through a given point.

There are sixteen twisted cubics which pass through four given points, have a given chord, have a tangent through a given point, and touch a given plane.

There are sixteen twisted cubics which pass through four given points, have a given chord, and have a tangent through each of two given points.

§ 7. We shall determine the number of curves k^3 of a surface σ of which one of the planes of osculation passes through a given straight line g. If we project the curves k^3 of σ out of one of the points of intersection G of g and σ on an arbitrary plane γ, the projections k'^3 form a pencil of cubics which have a double point in the intersection D of γ with the generatrix d of σ through G that belongs to the same scroll as l and which have single points in the intersection of γ with the other straight line of σ through G and in the projections of $H_1, \ldots, H_4$ out of G on γ. This pencil contains five degenerations. Such

a degeneration consists of the conic through four of the single base points and D and the join of D and the fifth single base point.

The locus of the points where curves of the pencil touch the straight lines through the intersection Q of g and γ, is a curve k^5 of the fifth order as a straight line through Q is touched by four curves and the locus contains Q. As there are two curves which have a cusp in D, the projections of the two curves k^3 of σ which touch d, and as there is at the same time a curve which touches the line QD at D, k^5 has a triple point in D. The branch of k^5 corresponding to the latter curve evidently touches this curve and, therefore, also the line QD at D. Further k^5 passes through the five single base points and the tangent at such a base point to k^5 passes through Q.

As the triple point D counts for three double points, k^5 has the class $5\times 4-3\times 2=14$. Two of the fourteen tangents to k^5 through Q coincide with the tangent at Q, five pass through the single base points of our pencil and one joins Q to the common double point. Accordingly there remain six tangents for each of which two of the curves k'^3 which touch it, coincide. These are inflectional tangents of curves k'^3 through Q. Consequently the inflectional tangents of the curves of our pencil envelop a curve of the sixth class [1]).

Now an inflectional tangent through Q is the intersection with the plane of projection of a plane of osculation through g to a curve k^3 on σ. We see, therefore, that a surface σ contains six curves k^3 of which a plane of osculation passes through a straight line g given at random, and that, consequently, the image curve of the system of the twisted cubics of our congruence which have a plane of osculation through g and of which the developable surfaces, therefore, touch g, is a curve g^6 of the sixth order.

The numbers of points of intersection of g^6 with a conic m^2, k^2, a curve b^4, a curve p^4, and another curve g^6 lead resp. to:

There are twelve twisted cubics which pass through four given points, have a given chord, cut a given straight line and have a plane of osculation through a given straight line.

There are twelve twisted cubics which pass through four given points, touch a given straight line and have a plane of osculation through a given straight line.

There are 24 twisted cubics which pass through four given points, have a given chord, touch a given plane and have a plane of osculation through a given straight line.

There are 24 twisted cubics which pass through four given points, have a given chord, have a tangent through a given point, and a plane of osculation through a given straight line.

There are 36 twisted cubics which pass through four given points, have

[1]) In this way the same class has been determined for a general pencil of cubics by Prof. H. J. VAN VEEN, Nieuw Archief voor Wiskunde, 2e reeks, dl. XII, p. 284.

a given chord and have a plane of osculation through each of two given straight lines.

§ 8. Now the system M of the ∞^4 conics m^2 of § 3 will be investigated more closely. Besides the system $\triangle$ of the ∞^3 pairs of lines which we found by combining the straight lines of α with the tangents to k^2, M contains four more systems of ∞^3 pairs of lines. For if m approaches a straight line m_1 through H_1 and if we consider the curves k^3 which cut m, the curves k^3 of which the points of intersection with m do not approach H_1, finally form the surface σ_1 which contains m_1, whereas the curves k^3 of which the points of intersection with m approach H_1, become those curves which touch at H_1 the limitary position μ_1 of the plane Hm. The locus of the latter curves k^3 is the surface σ'_1 which touches μ_1 at H_1. Now for μ_1 we may choose any plane through m_1. We find accordingly that to a surface σ_1 there correspond those surfaces σ'_1 which touch the line m_1 of σ_1 through H_1 belonging to the same scroll as l, at H_1 and which form, therefore, a pencil that contains σ_1 of which the base curve consists besides of l of the curve k^3 which touches m_1 at H_1.

The ∞^3 pairs (σ_1, σ'_1) are represented on ∞^3 pairs of rays (k_1, k'_1) of α which form degenerate conics of M. To a straight line k_1 there corresponds a plane pencil of lines k'_1 which has a point K_1 on k_1 as vertex. To a pencil of surfaces σ'_1 there corresponds one surface σ_1, the one which contains the tangent at H_1 to the cubic part of the base curve of the pencil. Consequently to a point K_1 there corresponds one line k_1. The line elements (k_1, K_1) form, therefore, a *bilinear null system* [1]) N_1, which we get by associating to any line k_1 of α the image point of the curve k^3 which touches the line m_1 through H_1 of the surface σ_1 corresponding to k_1, at H_1. It is easily seen that the lines t_2, t_3 and t_4 and the points T_{34}, T_{24}, and T_{23} are singular for N_1.

In this way we find a system $\triangle_1$ of ∞^3 degenerations of M consisting of the pairs of rays which are formed by the lines k_1 of α and the straight lines through the null point K_1 of k_1. In the same way it appears that to the points H_2, H_3, and H_4 there correspond three systems $\triangle_2$, $\triangle_3$, and $\triangle_4$ of degenerations of M which may be derived from null systems N_2, N_3, and N_4.

M also contains ∞^3 degenerations in pairs of points. The surfaces σ which touch an arbitrary line m, are represented on the tangents to the conic m^2 corresponding to m. If we choose m in the tetrahedral complex C^2, the surfaces σ touching m form two pencils which have the two curves k^3 that touch m, as base curves so that m^2 degenerates into the pair of points which consists of the image points of the two above mentioned curves k^3. The carrier of this pair of points is the image of

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1) Cf. Prof. JAN DE VRIES. *Plane Linear Null Systems*, these Proceedings **15**, 1165.

the surface σ which contains m. Accordingly an arbitrary line k of α to which the ∞^1 lines m of a scroll of the surface σ corresponding to k are associated, contains ∞^1 of such pairs of points. As a curve k^3 of σ is touched by two generatrices of the said scroll, and, therefore, touches the same generatrix as two other curves k^3 of σ, the pairs of points of a line k of α which give degenerations of M, form an involutory (2, 2)-correspondence on k. Any pair of points of one of the lines $t_1, \ldots, t_4$ is a quadruple pair of M, as two conics in one of the planes $\beta, \ldots, \beta_4$ have four common tangents.

Let us now determine the number of the conics m^2 that pass through four given points α. To them correspond the four lines m which cut in different points the four curves k^3 of which the given points are the images.

Let us first consider the surface of the lines which cut a given straight line r and two curves k^3 in different points. As through any point of r there pass five generatrices of this surface and a plane through r contains nine generatrices, this surface is of the order 14. The plane through H_k and r contains four generatrices through H_k two of which cut one curve and two of which cut the other one outside H_k. We find one more generatrix through H_k, if we cut the common plane of contact of the two curves at H_k with r and if we join the point of intersection to H_k. Consequently each point H_k is a five-fold point of the surface which cuts a third curve k^3 outside the points H_k, hence in $3 \times 14 - 4 \times 5 = 22$ points.

The surface of the lines which cut three curves k^3 in different points, is, therefore, of the order 22. The generatrices of this surface through H_k are in the first place three lines each of which cuts two of the curves k^3 in separate points different from H_k. Further any plane of contact of two curves k^3 contains two generatrices which cut the third curve k^3 outside H_k. Accordingly we find that any point H_k is a nine-fold point of the surface and that a fourth curve k^3 cuts this surface outside the points H_k in $3 \times 22 - 4 \times 9 = 30$ points. There are, therefore, thirty lines which cut four curves k^3 in different points. Among them the line l, which cuts each of the curves k^3 twice, counts 16 times. As many conics m^2 passing through the four given points are associated to the other 14 lines.

The result found above may also be derived by the aid of the well known property that the number of conics of a system of ∞^1 individuals which pass through a given point, is equal to $^1/_3$ of the number of degenerations into two straight lines which are contained in the system if this system does not contain any degenerations into pairs of points. The latter is the case with the system T of the conics of M which pass through the three given points A_1, A_2, and A_3 of α. To T there belong six pairs of lines of Δ, e. g. those which are formed by the lines $A_2 A_3$ and the two tangents through A_1 to k^2. If we combine

$A_2 A_3$ and the line which joins A_1 and the null-point K_1 of $A_2 A_3$, we get a pair of lines of $\triangle_1$ which belongs to T. As the null points K_1 of the lines through A_1 form a conic which cuts $A_2 A_3$ in two points, there are two more pairs of lines of $\triangle_1$ belonging to T, which consist of $A_2 A_3$ and a straight line through A_1. Accordingly 9 pairs of each of the systems $\triangle_1, \ldots, \triangle_4$ belong to T so that in all T contains 42 pairs of lines. Hence 14 conics of M pass through four given points.

If we associate to each other the points of intersection of each conic of T with an arbitrary straight line, a (14, 14)- correspondence arises on this straight line, so that, as T does not contain any double lines, M contains 28 conics which pass through three given points and touch a given straight line.

This appears also if we remark that to the conics that satisfy the above mentioned conditions, there correspond lines m which cut three given curves k^3 in different points and which touch a surface σ. The surface of the order 22 of the lines which cut three curves k^3, has 44 lines in common with the quadratic line-complex of σ. Among them l, which is a double line of the complex of tangents, counts 16 times, so that in fact there remain 28 lines.

The congruence (5, 9) of the lines which intersect two curves k^3, has 56 lines in common with the congruence (4, 4) of the tangents to two surfaces σ, among which l again counts 16 times. There are, therefore, 40 conics m^2 which touch two given straight lines and pass through two given points.

The scroll of the common tangents to three surfaces σ, which is of the order 16, contains 48 lines which intersect a curve k^3. If again we set apart l 16 times, it appears that there are 32 conics m^2 which pass through a given point and touch three given straight lines.

Finally we find that four surfaces σ have 32 common tangents, if we count l 16 times. Consequently there are 16 conics m^2 which touch four given straight lines.

Physics. — *"The thermal expansion of copper between + 101° C. and — 253° C."* By W. H. KEESOM, F. P. G. A. J. VAN AGT and Miss A. F. J. JANSEN. (Communication N°. 182*a* from the Physical Laboratory at Leiden).

(Communicated at the meeting of February 27, 1926)

§ 1. *Introduction.* As a continuation of the investigations of the thermal expansion of solid substances at low temperatures, which have been made in this laboratory by KAMERLINGH ONNES and HEUSE (Jena-glass 16^{III} and Thuringian-glass to — 182° C.) [1]), by KAMERLINGH ONNES and CLAY (Jena-glass 16^{III} and platinum to — 182° C.) [2]) and by VAN AGT and KAMERLINGH ONNES (Jena-glass 16^{III} to — 253° C.) [3]), we have investigated that of copper to the temperature of liquid hydrogen boiling under atmospheric pressure.

§ 2. *Method and apparatus.* We followed the method given by VAN AGT and KAMERLINGH ONNES (l. c.). A vacuumvessel about 1 metre long (see fig. 1) was constructed from electrolytic copper by L. OUWERKERK, "technician 1st class" at the Physical Laboratory. The inner wall of this vacuumvessel was brought to different temperatures by pouring into it successively water at about room temperature, boiling water, and liquefied gases. Bent platinum points B, C, B', C' were fixed to the bottom and to the top of the inner and outer tubes, and the variations of the distances between them were measured by the method described in Comm. N°. 176*a*. The outer wall was kept at constant temperature by means of a waterjacket. The vacuum was produced and maintained by means of charcoal in liquid air, the charcoaltubes N_1 and N_2 being connected in turn through the taps P_3, P_1 and P_2. Charcoal was also placed into the vacuum space at O. For the measurement with liquid hydrogen tap P_3 was closed. Owing to the greater conductivity of copper, compared to that of glass, the liquefied gas, brought in it, evaporated appreciably quicker than with the measurements of Comm. N°. 176*a*. The vacuumvessel was therefore provided with a widened part R, made of new-silver, so that it could hold a larger quantity of liquid.

The kathetometers, provided with micrometer-eyepieces, which served for observing the platinum points, now carried a lamp with reflector to illuminate the field of view. A standard-metre was mounted near the tube, so that the standard-metre and points could be seen sharply with

1) These Proceedings 7, 674, 1905, Comm. Leiden N°. 85.

2) These Proceedings 9. 199 1906, Comm. Leiden N°. 95*b*.

3) These Proceedings 28. 667, 1925, Comm. Leiden N°. 176*a*. Comp. F. P. G. A. J. VAN AGT, Thesis for the Doctorate Leiden, p. 61.

the same adjustment of the telescope. The magnification of the microscopes was determined by measuring, before each measurement of the distance of the platinum points, some mm.'s of the standard-metre.

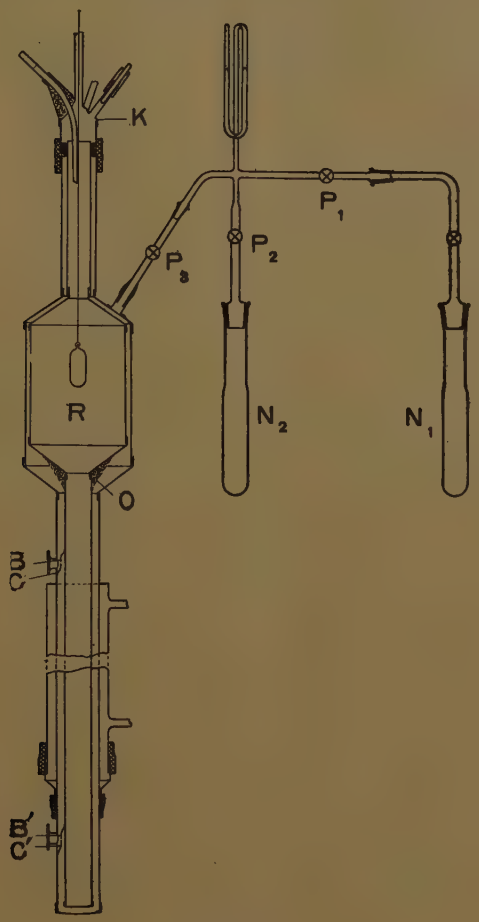

Fig. 1.

The distance of the points was determined repeatedly at room temperature. The temperatures of the inner and outer walls were read off with mercury thermometers, graduated in tenths of a degree. Using the value, found by FIZEAU, for the expansion coefficient at this temperature, this result was reduced to the temperature of 0° C. for both inner and outer walls. The measurements at room temperature were repeated before and after each measurement at low temperatures in order to ascertain that the distance between the points had regained its original value, and that the points had not been distorted; for, during the filling with liquid hydrogen, the tube bent appreciably.

Besides those at room temperature, measurements were made at the boiling-points of water, ethylene, oxygen and hydrogen.

In order to obtain a temperature of about 100° C., boiling distilled

water was poured out into the tube, and then a heating coil introduced to keep the water boiling. The temperature of the water at the top was derived from the reading of the barometer; that of the tube at the bottom by taking into account the hydrostatic pressure of the water column; the average of both temperatures being taken. The temperature of the hydrogen bath was determined in the same way. In this case the complementary pressure in the gasholder was added to the height of the barometer.

The temperatures of the ethylene and of the oxygen baths were determined by means of the resistance thermometer Pt_{32}, which had been calibrated several times with the helium thermometer, and which was placed half-way between the platinum-points.

Before the tube was filled with a liquefied gas, it was cooled by means of liquid air. This appeared to be particularly necessary in the case of hydrogen, otherwise too much of it had to be used to cool the tube. The tube was kept for some moments filled with liquid air, and then the air was quickly siphoned out. After this the new-silver cap K was put on, and connected with the vacuum-lead in order to evacuate the tube. The tube could now be filled with liquid hydrogen. A float was passed through the cap to indicate if the liquid stood in the tube, and its height.

§ 3. *Results.* After some provisional measurements, the results of Table I were obtained.

TABLE I.

Date	Temperature		$L_{inner}-L_{outer}$
	Inner tube	Outer tube	
Dec. 1 '25	reduced to 0° C.	reduced to 0° C.	0.129 mm.
„ 17 '25	„	„	0.120 „
„ 22 '25	—252.84° C.	„	—2.600 „
„ 23 '25	reduced to 0° C.	„	0.158 „
Jan. 5 '26	„	„	0.149 „
„ 6 '26	„	„	0.153 „
„ 18 '26	„	„	0.155 „
„ 19 '26	„	„	0.159 „
„ 21 '26	—102.87° C.	„	—1.342 „
Febr. 3 '26	reduced to 0° C.	„	0.157 „
„ 4 '26	—185.47° C.	„	—2.286 „
„ 10 '26	reduced to 0° C.	„	1.600 „

The platinum-points were distorted somewhat before the measurement at 100° C. The following measurements were then made.

TABLE I (Continuation).

Febr. 17 '26	reduced to 0° C.	reduced to 0° C.	0.782 mm.
„ 18 '26	100.94° C.	„	2.323 „
„ 19 '26	reduced to 0° C.	„	0.772 „

From the above the following changes of length were derived:

From 0° C. to — 102.87° C. contraction 0.155 + 1,342 = 1.497 mm.
„ — 102.87° C. to — 185.47° C. „ 0.944 „
„ — 185.47° „ „ — 252.84° „ „ 0.314 „
„ 0° C. „ — 100.94° „ expansion 1.546 „

Length of the inner tube: 947.96 mm.

The change of length as a function of the temperature is represented in Fig. 2.

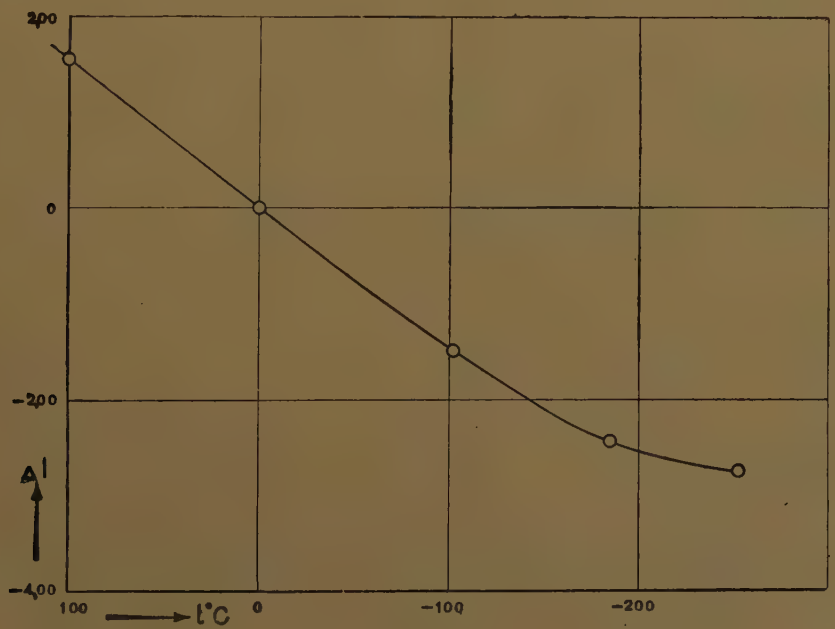

Fig. 2.

§ 4. *The expansion coefficient.* From the results the following mean expansion coefficients were calculated.

temperature range	$\bar{\alpha}$
between 100.94° C. and 0° C.	16.16 $\times 10^{-6}$
0 ° C. „ — 102.87° C.	15.35 „
— 102.87° C. „ — 185.47° C.	12.10 „
— 185.47° C. „ — 252.84° C.	4.92 „

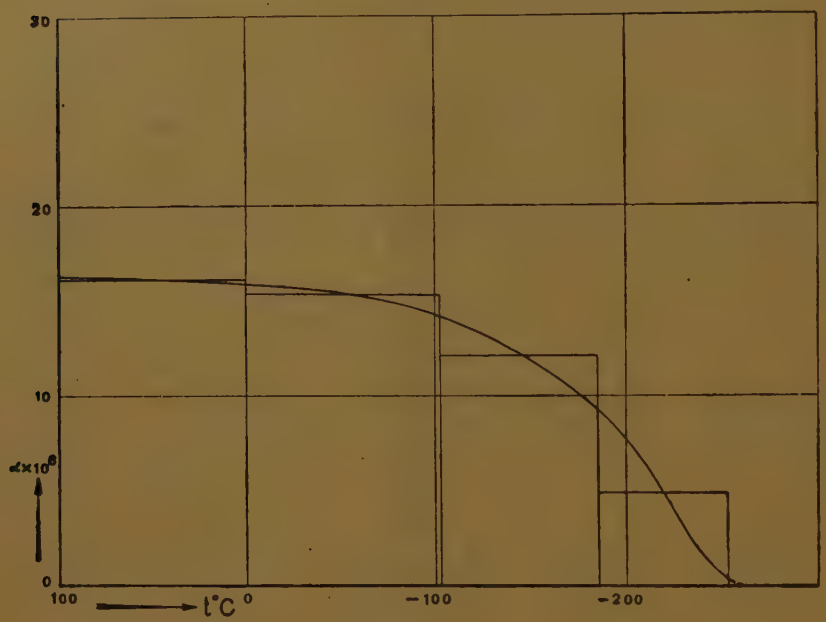

Fig. 3.

In Fig. 3 we derived graphically the expansion coefficient $\alpha = \frac{1}{l}\frac{dl}{dt}$ by drawing a smooth curve, so that, for the temperature ranges mentioned above, $\int_{t_1}^{t_2} \alpha\, dt = \bar{\alpha}\,(t_2 - t_1)$. The equality of the areas in question was tested with the aid of a planimeter.

§ 5. *Representation of the observations in a formula.*

It was not possible to represent the observations over the whole range from + 100.94° C. to — 252.84° C. by a third degree equation. Therefore, in analogy to the formula for the expansion of glass [1]), a fourth-degree equation was made of the form:

$$L_t = L_0\left[1 + \left\{a\frac{t}{100} + b\left(\frac{t}{100}\right)^2 + c\left(\frac{t}{100}\right)^3 + d\left(\frac{t}{100}\right)^4\right\} . 10^{-6}\right].$$

[1]) F. P. G. A. J. VAN AGT and H. KAMERLINGH ONNES, l. c. p. 786 note 3.

From our observations we calculated the coefficients:

$$a = 1585.338$$
$$b = 18.121$$
$$c = -8.698$$
$$d = 20.662$$

§ 6. *Comparison with previous observations.* Previous investigations on copper at low temperatures were made by DORSEY [1]), following the method of FIZEAU, in the temperature range from + 10° C. to — 170° C.; by HENNING [2]), who measured between — 191° C. and + 16° C. the expansion relative to platinum [3]); by CH. L. LINDEMANN [4]), who measured down to — 253° C. the expansion relative to fused quartz, and by BORELIUS and JOHANSSON [5]), who measured the expansion of copper relative to quartz [6]), between 0° C. and — 170° C.

The measurements of LINDEMANN do not allow a direct comparison as the measurements are relative to fused quartz, of which the expansion coefficient is not known. [7])

For the other observations we have calculated from their results the mean expansion coefficients for the temperature ranges, mentioned in Table II, and compared these coefficients with ours:

TABLE II.

Mean expansion coefficient × 10^6				
Temperature-range	DORSEY	HENNING	BORELIUS and JOHANSSON	KEESOM, v. AGT and Miss JANSEN
0° C. to —100° C.	15.63		15.4	15.36
0° „ „ —180° „	14.24	14.22	14.0	14.01
—100° „ „ —180° „	12.51		12.2	12.35
—180° „ „ —250° „	—		—	5.45

1) H. G. DORSEY. Phys. Rev. 25, 88, 1907.

2) F. HENNING. Ann. d. Phys. 22, 631, 1907.

3) The expansion coefficient of platinum in this range was known from measurements made by SCHEEL, Verh. d. D. Physik. Ges. 9, 3, 1907, by the method of FIZEAU.

4) CH. L. LINDEMANN. Physik. Z. S. 12, 1197, 1911.

5) G. BORELIUS and C. H. JOHANSSON. Ann. d. Phys. (4) 75, 23, 1924.

6) For quartz (the authors do not mention what sort of quartz they used) use was made of the measurements of H. G. DORSEY, Phys. Rev. 27, 1, 1908.

7) Comp. H. G. DORSEY, note 1.

Physics. — *"Thermionic Emission of the Metals Tungsten, Molybdenum, Thorium, Zircon and Hafnium"*. By C. ZWIKKER. (Communicated by Prof. P. ZEEMAN.)

(Communicated at the meeting of February 27, 1926).

1. When the thermionic emission of the metals is considered as an evaporation of the electron gas from the metal, it can be thermodynamically deduced in a simple way that the saturation current per square centimeter may be represented by the following formula [1]):

$$i = AT^{\frac{c_p}{R}-\frac{1}{2}} e^{-\frac{\lambda_0}{RT}-\int_0^T \frac{\varepsilon}{RT^2}dT} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where:

A is a constant.

c_p = heat capacity of 1 gr. mol. of electrons outside the metal.

λ_o = external heat of evaporation of 1 gr. mol. of electrons at the absolute zero point.

ε = energy content of 1 gr. mol. of electrons found inside the metal at the temperature T.

As the "electron gas" may certainly be considered as mon-atomic, we may put $c_p = \frac{5}{2}R$, so that the exponent of T becomes equal to 2 in the above formula.

When also ε is neglected, we arrive at RICHARDSON-DUSHMAN's simple formula:

$$i = AT^2 10^{-\frac{b}{T}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which A and b are constants of material.

The same result is arrived at when ε is put independent of the temperature, because it is then included in the b. Consequently the variation of the energy content of the electrons with the temperature, i.e. the THOMSON effect, is neglected in both cases.

By application of SACKUR and TETRODE's theory of the chemical constant, DUSHMAN [2]) has also been able to calculate the constant A quantitatively. It then appeared that A is independent of the nature of the material, and is equal to:

$$A = \frac{2\pi e m k^2}{h^3},$$

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[1]) Cf. e.g. v. RASCHEVSKY, Z. f. Phys. 32, 1925, p. 746.

[2]) Phys. Rev. 21, 1923, p. 623.

in which e, m, k, and h have the usual meaning of the fundamental constants. When i is expressed in amperes per cm^2, then $A = 60{,}2$. V. RASCHEVSKY [1]) arrives at the same value of A by means of a quanta-statistic calculation, however, again if the THOMSON effect is neglected.

Introducing the THOMSON effect, we must, therefore, calculate the integral

$$\int_0^T \frac{\varepsilon}{RT^2}\, dT \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

A first condition will be that at the absolute zero ε be $= 0$, and also $\frac{d\varepsilon}{dT}$, because otherwise the integral becomes infinite.

When now, in order to find the order of magnitude of the correction of the saturation current ensuing from the THOMSON effect, we put the energy content proportional to the temperature, hence

$$\varepsilon = Ne\sigma T,$$

in which, therefore, the THOMSON coefficient σ is taken to be independent of the temperature, and N represents the number of AVOGADRO, we have probably taken ε too great, and with even greater probability the integral (3) likewise too great.

This assumption for ε gives rise to the appearance of the factor

$$T^{-\frac{Ne\sigma}{R}}$$

in the formula for i.

$\frac{Ne\sigma}{R}$ is of the order $\pm 0{,}1$, the factor $T^{-\frac{Ne\sigma}{R}}$ is of the order $\frac{1}{2}$ or 2, and almost independent of the temperature. If nevertheless we wish to determine the emission by means of a formula of the form:

$$i = AT^2\, 10^{-\frac{b}{T}},$$

the quantity A will include the almost constant factor $T^{-\frac{Ne\sigma}{R}}$, i.e. the exact value of A need not be 60.2, but can be a few times larger or a few times smaller. For tungsten for instance, σ is known at high temperatures, viz.:

$$-2.10^{-5}\ \text{Volt/degree}, \qquad e\sigma = -3{,}184\,.\,10^{-24}\ \text{JOULE}$$

$$k = \frac{R}{N} = 13{,}72\,.\,10^{-24}\ \text{JOULE} \quad \text{hence} \quad \frac{e\sigma}{k} = -0{,}232.$$

Consequently we expect for tungsten the formula:

$$i = 60{,}2\, T^{2+0{,}232}\,.\,10^{-\frac{5}{T}}$$

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[1]) l. c.

which for a mean temperature of 2000° almost coincides with:

$$i = 355\, T^2\, 10^{-\frac{b}{T}}.$$

Tungsten has an exceedingly great THOMSON coefficient. For most other metals A will depart less from the universal value 60.2. Experimentally it can, however, scarcely be decided whether the factor 355 represents the measurements better than the factor 60.2, because a small variation of the quantity b will bring the absolute value of the emission in harmony with the formula.

An attempt to find experimentally something about the value of the quantity A, has been made by DUSHMAN, ROWE, EWALD and KIDNER [1]), by measuring the emission of tungsten, molybdenum and tantalum. The result at which they arrived, was that it was not to be proved experimentally that the value 60.2 is wrong.

The writer of this paper has made emission measurements of some metals which enable him to make an investigation of the constant A.

2. Measurements of the very pure tungsten have been published by me in detail elsewhere [2]).

The mean of the measurements on 5 filaments was:

T °K	i Amp./cm.^{2}
2500	0.324
2400	0.119
2300	0.0425
2200	0.0140
2100	0.00398
2000	0.000978

For each of these temperatures we can calculate with the formula:

$$i = A T^2\, 10^{-\frac{b}{T}}$$

what b we must fill in when we put $A = 60.2$. We can do this too for any value of A and will, accordingly prefer that A for which the six values for b are as much as possible equal to each other. I have carried out this calculation for the 5 values of A: 10; 60.2; 100; 355; and 1000, with the following results:

[1]) Phys. Rev. 25, 1925, p. 338.

[2]) Dissertatie Amsterdam 1925. Archives Néerlandaises IX. 1925, p. 301.

Value of b:

T	$A=10$	$A=60.2$	$A=100$	$A=355$	$A=1000$
2500	20740	22670	23200	24590	25730
2400	20850	22730	23230	24600	25640
2300	20980	22720	23230	24490	25570
2200	20990	22730	23200	24380	25420
2100	21120	22770	23200	24390	25320
2000	21238	22780	23230	24320	25220

These calculated values have been drawn in figure 1. It is seen at a glance that the value $A = 100$ gives the most constant value for b. For $A = 60.2$ the values of b diverge 11 on 2270. In the formula occurs the quotient of b and T. In the temperature a difference of 11° at 2270° can certainly be measured. Estimating the error in the temperature at 5°, that in the emission measurement at 5 %, once more corresponding to an error in the temperature of 5°, we see that the experimental error is 10°. Unless we underrate the accuracy of the (already averaged) measurements, it follows from this that it cannot be concluded from the measurements of tungsten that the value 60.2 for A is really wrong. It *is*, however, certain that the value $A = 355$, which would suppose an error of 27° in the temperature measurement, is inadmissible. This latter is probably caused by our having introduced a too high amount for the integral (3).

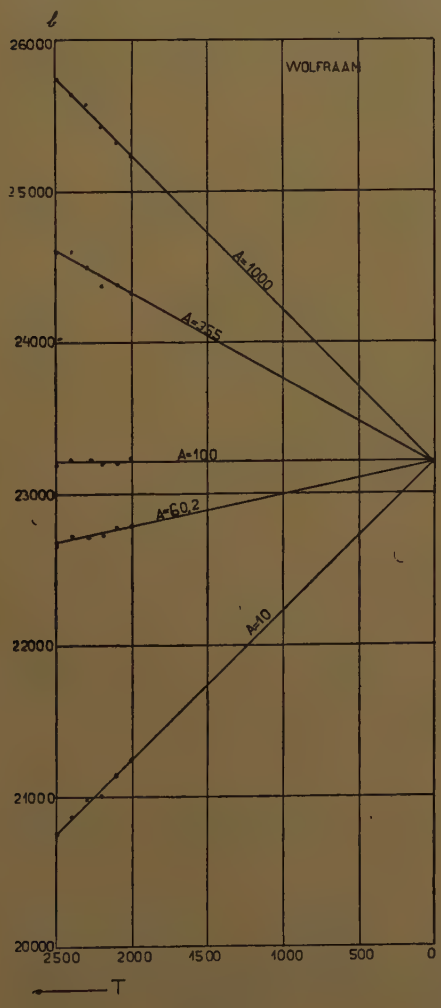

Fig. 1.

3. Emission measurements of molybdenum have been performed in tubes of the model represented in fig. 2 by receiving the electron emission of the central incandescent filament on a spiralized anode. The

course of the measurement was the same as that for tungsten [1]) with this difference that for tungsten the right emission is only obtained when the lamp is immersed in liquid air, whereas this appeared to be unnecessary for molybdenum. Part of the glass wall was, however, covered with magnesium to maintain the vacuum. The emission has been measured for 2 different kinds of filaments. The first filament was supplied by PHILIPS, the second was an American filament of the General Electric Company. The incandescent-temperatures have been determined with an optical pyrometer. The brightness-temperature is measured. To reduce this to true temperature the emissive power of molybdenum is required. I have taken for this the values given by WORTHING [2]).

Fig. 2.

The two kinds of filaments gave almost the same results. The PHILIPS filament gave slightly greater emission, about 10 %. The deviation inter se of the two filaments was always less than 10° in temperature.

After taking the mean of the two filaments, and after a correction had been applied for the SCHOTTKY effect [3]) the relation between emission and temperature was:

T °K	i Amp./cm.^{2}
1900	0.000500
2000	0.00234
2100	0.00859
2200	0.0285
2300	0.0835
2400	0.236

We now repeat the calculations we have also performed for tungsten, and find the following table for b as function of A and T. (See Table following page).

These points are given in fig. 3. We see from this figure that the most probable value of A is: 65. There remains, however, some licence in the figure as regards the tracing of the lines $A =$ constant. The value 100 for A is not excluded.

4. Dr. DE BOER put some strips of metallic thorium at my disposal, made by decomposition of gaseous thorium compounds at an incandescent

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[1]) l. c.

[2]) Phys. Rev. 25, 1925, p. 846.

[3]) By SCHOTTKY effect is understood the phenomenon that the saturation current still increases slightly with the tension. The emissions mentioned here are those which the filament would give in the absence of anode tension.

tungsten filament[1]). If this tungsten filament diffuses through the thorium, when the thorium is aged, the thorium strips used can contain about 6 $^0/_{00}$ of tungsten.

Value of b in molybdenum.

T	$A=10$	$A=100$	$A=1000$
2400	20140	22530	24900
2300	20250	22560	24860
2200	20300	22510	24710
2100	20370	22430	24550
2000	20470	22460	24460
1900	20630	22530	24420

The electron emission of these strips was very constant and reproducible, when only in the tube a magnesium mirror was adjusted. Definitive measurements have been made with a strip of a cross-section of 0.110×0.0090 cm. and a length of 4.23 cm. According to estimation 1 mm. was lost at each end of the strip, because it had not fully the required temperature. If this estimation is 100 $^0/_0$ wrong, this gives in the emission per cm^2 an error of no more than 5 $^0/_0$.

The temperatures have again been optically determined. The recalculation of brightness temperature to true temperature cannot take place so accurately as in tungsten or molybdenum, because little is known about the emissive power. I have put the emissive power at 0.40.

With these data the relation between temperature and emission corrected for the SCHOTTKY effect becomes:

T °K	i Amp./cm.^{2}
1300	0.0000133
1400	0.000100
1500	0.000738
1600	0.00430
1700	0.0203
1800	0.0794

This emission may be expressed in the formula:

$$i = AT^2 10^{-\frac{b}{T}},$$

[1]) Z. f. Anorg. u. Allgem. Chemie 148, 1925, p. 345.

when b is taken as follows:

T	$A = 10$	$A = 100$	$A = 1000$
1300	15770	17060	18350
1400	15800	17210	18610
1500	15720	17220	18730
1600	15630	17220	18820
1700	15570	17270	18950
1800	15500	17310	19120

Cf. fig. 4. As most probable value of A follows 70. It is, however, the question in how far this result is reliable in connection with the unknown emission coefficient of thorium. I have taken 0.4 for this. If in reality it is 0.5, my temperatures have been taken too high. The error is 20° at 1300° and 32° at 1800°. The relative error is 12° to an average of 1500°. The same relative error occurs in b. Then the most probable value for A becomes 40. It is, however, not probable that the emissive power is so far wrong, as was supposed here.

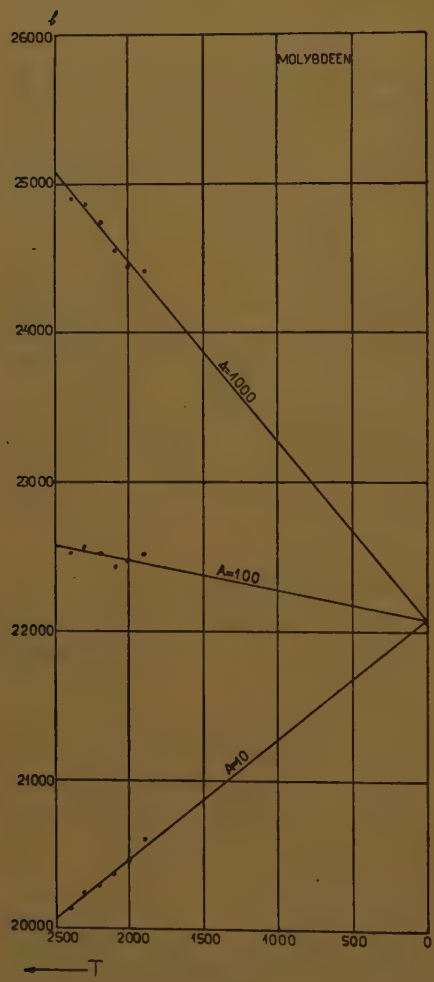

Fig. 3.

5. Also Dr. DE BOER supplied me with a zircon filament which he had prepared on a tungsten nucleus by decomposition. The nucleus was 40 microns, the zircon filament 770 microns, so that the tungsten content was 4.8 $^0/_{00}$ in weight. Of this filament I have made three tubes with incandescent filament lengths resp. 8.85, 12.60, and 16.45 cm. By subtracting the emissions of these filaments from each other, I was enabled to eliminate the cooled extremities. In order to convert the measured brightness temperature into true temperature the emissive power has been used, as given by BURGESS

and WALTENBERG [1]), viz. 0.32. The electron emission of zircon appeared to be little sensitive to gas rests, greatly in contrast to that of tungsten. The emission corrected for the SCHOTTKY effect was as function of the temperature:

T °K	i Amp./cm.^{2}
1600	$5.06.10^{-5}$
1700	$3.96.10^{-4}$
1800	$2.29.10^{-3}$
1900	$1.18.10^{-2}$
2000	$5.50.10^{-2}$
2100	$1.97.10^{-1}$

Calculating the relation between A and b with these data, we get the following table for b:

T	$A=10$	$A=100$	$A=1000$	$A=10000$
1600	18720	20340	21930	23550
1700	18460	18460	21860	23570
1800	18270	18270	21860	23670
1900	18030	18030	21820	23690
2000	17720	17720	21730	23720
2100	17530	17530	21750	23820

Compare fig. 5. We see that it is out of the question that 60.2 could be the correct value for A. The most probable value for A is according to the figure: $A = 3000$.

6. Finally we can derive the value of A from measurements with a hafnium filament, which has also been made by Dr. DE BOER. These measurements bear a preliminary character, because:

1. the material is not quite pure; it contains 5 % of tungsten in consequence of the preparation on a tungsten nucleus, it not being excluded either that it contains a few percentages of zircon, which is difficult to detect by the side of hafnium.

2. there is little known about the losses at the cooled extremity of the filament.

3. the emission coefficient is not known.

[1]) Bull. Bur. of St. 11, 1915, p. 591.

Yet the measurements yield a value for A, which departs so much from the theoretical value, that it should certainly be mentioned in this connection.

With regard to the losses at the ends, I have proceeded as if it were zircon. For the emission coefficient I have assumed the estimated value 0.37.

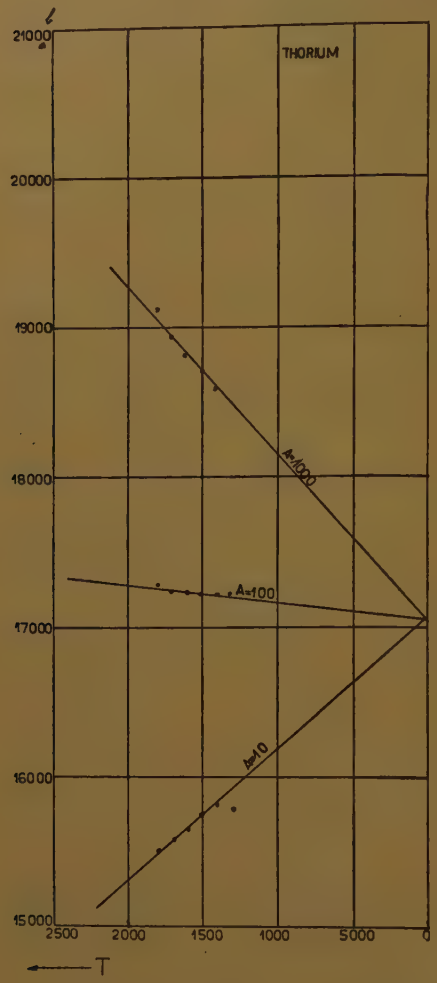

Fig. 4.

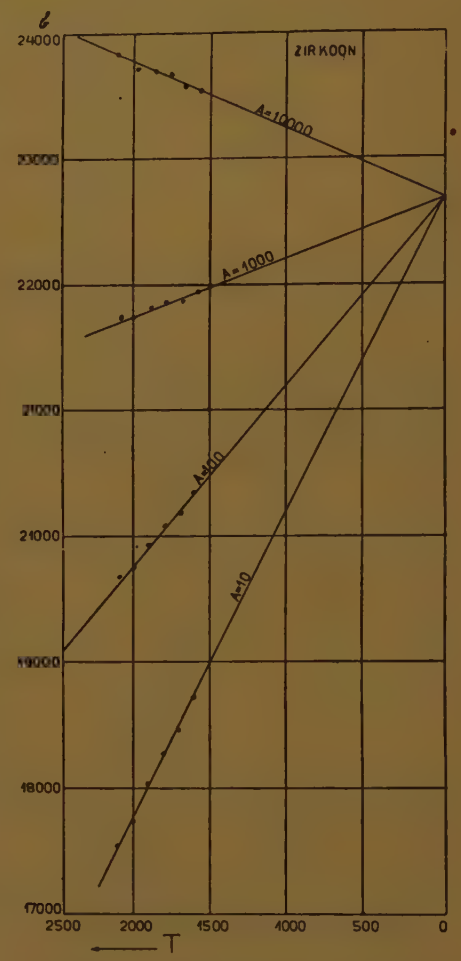

Fig. 5.

The measurements above 1900° (under 1900° not yet explained irregularities occurred) yielded the result.

T °K	i Amp./cm.^{2}
1900	$4.80.10^{-3}$
2000	$2.62.10^{-2}$
2100	$1.23.10^{-1}$
2200	$4.85.10^{-1}$

The corresponding values for A and b are:

T	$A=100$	$A=1000$	$A=10000$	$A=100000$
1900	20620	22520	24420	26340
2000	20350	22350	24330	26350
2100	20070	22160	24250	26390
2200	19810	22000	24200	26410

From figure 6, in which these points are indicated, it appears that the most probable value for A is: 55000.

7. Summarizing we may, therefore, say that the metals tungsten, molybdenum, thorium, and according to DUSHMAN c. s. also tantalum yield a value for A which is in accordance with the theoretical value, perhaps slightly modified by the THOMSON effect. The values which zircon and hafnium give, depart however greatly from it.

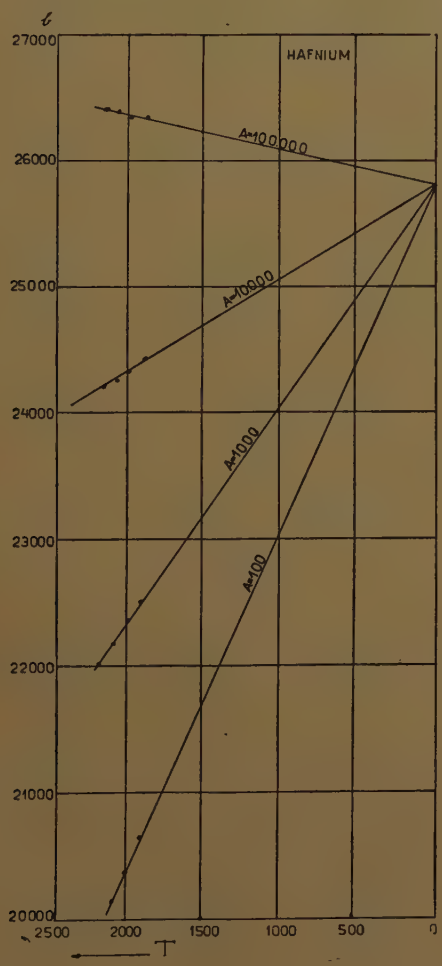

Fig. 6.

Also DUSHMAN c. s. observe that greatly divergent values of A do occur. In this case there is, however, always question of a thin (sometimes mono-molecular) layer of emitting substance on a metal filament. For instance in the emission of a tungsten filament covered with cesium A is smaller than 0.003 [1]), and in the emission of a superficially oxidized tungsten filament A is $= 5.10^{11}$. Without giving a further explanation of it, DUSHMAN c. s. state that A will always become greater than the theoretical value when the covering layer is electronegative.

When we adopt this view we might account for the high values which zircon and hafnium give for A by assuming that they are covered up to 2200° K (melting-point of zircon) by an electronegative layer. In favour of the view that zircon

1 KINGDON, Phys. Rev. 1924, p. 510.

at low temperatures is covered with an oxide layer, speaks its unsusceptibility to mineral acids, a property that it shares with aluminium and magnesium, of which it is almost certain that they are covered with a coat of oxide. If it is, besides, taken into account that zircon oxide melts higher than zircon itself, and probably the velocity of evaporation of zircon-oxide is smaller than that of zircon, it is very well probable that zircon remains covered up to its melting-point. A same reasoning applies to hafnium. In this case we should, so far, have no example as yet of a pure metal surface the emission of which could not be represented by:

$$i = 60.2\ T^2\ 10^{-b/T}\ Amp/cm^2.$$

Mathematics. — *"On the Geometry of the Group-manifold of simple and semi-simple groups"*. By E. CARTAN and J. A. SCHOUTEN. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of March 27, 1926).

If the ∞^r transformations of a continuous group with r parameters are represented by points of an X_r, then the group-manifold (variété du groupe, Gruppenmannigfaltigkeit) arises. The parameters play the part of coordinates of the X_r. In the present paper we will prove that there exist in X_r three connexions (connexions, Uebertragungen) two of them being not symmetrical but having zero curvature, the third being symmetrical (à torsion nulle) and in the case of a simple or a semi-simple group Riemannian. The Riemannian connexions, formed in the above manner, are of a very particular type, having, as far as we know, not yet been examined except the case $r=3$, where the curvature becomes constant. The main feature of these geometries is the possession of two absolute parallelisms, who in the case $r=3$ pass into the well known parallelisms of CLIFFORD. In a subsequent article we will prove that this property is characteristic for the geometries of semi-simple groups, but for one exception, which is in close connexion with the non-associative numbersystem with 8 unities of GRAVES-CAYLEY.

§ 1. *The first displacement with zero curvature* (—).

The transformations of the group being given by the equations

$$'x^k = x^k(x^1, \ldots, x^n, \xi^{a_1}, \ldots, \xi^{a_r}); \; k=1, \ldots, n, \quad . \quad . \quad . \quad (1)$$

the ξ^ν are coordinates in the group-manifold X_r.

T_ξ corresponding with ξ^ν and $T_{\xi+d\xi}$ with $\xi^\nu + d\xi^\nu$, the linear element $d\xi^\nu$ corresponds with the infinitesimal transformation

$$T_{\xi+d\xi}\, T_\xi^{-1} \; ^{1})$$

This infinitesimal transformation has not only significance in the point ξ^ν. In any other point η^ν it corresponds with the linear element, extending from T_ξ to $T_{\eta+d\xi}\, T_\xi^{-1}\, T_\eta$. By this correspondence, which may be characterized by the equation

$$T_{\xi+d\xi}\, T_\xi^{-1} = T_{\eta+d\eta}\, T_\eta^{-1} \; ^{2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

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1) This means that T_ξ^{-1} has to be executed first and then $T_{\xi+d\xi}$.

2) See note page 807.

a connexion in X_r is defined, and this connexion is integrable, since because of (2) with a definite linear element in any point corresponds one and only one linear element in any point of the neighbourhood. If this connexion, which we will represent by (—), were symmetrical, it would be possible to lay in X_r a Cartesian system of coordinates and X_r would be an E_r. We will, however, see that this is not the case in general.

T_u being the general transformation of a subgroup with one parameter, the equation

$$T_\xi = T_u\, T_{\xi_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

represents a geodesic through $\overset{\nu}{\underset{0}{\xi}}$. Writing (2) in the form

$$T_{\xi'}\, T_\xi^{-1} = T_{\eta'}\, T_\eta^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

this equation defines the (—)-aequipollence of two finite segments of geodesics.

Besides the system of coordinates ξ^ν we introduce in every point of X_r a system of coordinates ε^k, geodesic in that point and we place these systems in such manner that two elements with the same coordinates $d\varepsilon^k$ in different points arise from one another by a (—)-parallel displacement and correspond consequently with the same infinitesimal transformation. Then in every point

$$\frac{\partial' x^k}{\partial \xi^\lambda} = \frac{\partial' x^k}{\partial \varepsilon^j}\,\frac{\partial \varepsilon^j}{\partial \xi^\lambda} \quad [1] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

holds good. Now $\dfrac{\partial' x^k}{\partial \varepsilon^j}$ depends only on $'x^k$ and is independent of the ξ^ν because $d\varepsilon^k$ represents in every point the same infinitesimal transformation. $\dfrac{\partial \varepsilon^j}{\partial \xi^\lambda}$ depends only on the relative situation of the systems ξ^ν and ε^k, i.e. therefore on ξ^ν and is independent of the $'x^k$. Writing for these functions

$$\frac{\partial' x^k}{\partial \varepsilon^j} = \xi_j^k\,('x) \quad ; \quad \frac{\partial \varepsilon^j}{\partial \xi^\lambda} = \psi_\lambda^j\,(\xi)$$

we obtain

$$\frac{\partial' x^k}{\partial \xi^\lambda} = \xi_j^k\, \psi_\lambda^j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

ψ_λ^j being nothing else but the affinor of unity, with regard to ε^k above and to ξ^λ beneath, the rank of ψ_λ^j is r and the equation (5) admits therefore an inversion of the form

$$\xi_i^k = a_i^\lambda\, \frac{\partial' x^k}{\partial \xi^\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

So we have obtained the first part of the first fundamental theorem

[1] We omit all signs Σ with regard to two differently placed indices.

of LIE [1]) and we see that this theorem is in close connexion with the displacement (—).

Writing $e^\nu dt$ for $d\xi^\nu$ and consequently $e^k dt$ for $d\varepsilon^k$, e^ν is a (—)-constant vector in X_r, having consequently constant coordinates with regard to the systems ε^k. (5) gives, that $d\varepsilon^k$ corresponds with the infinitesimal transformation

$$d'x^k = \xi_j^k \, e^j \, dt$$

or

$$df = \xi_j^k \, e^j \frac{\partial f}{\partial' x^k} = e^j \, 'X_j f \, dt \quad ; \quad 'X_j f = \xi_j^k \, ('x) \frac{\partial f}{\partial' x^\nu}.$$

With e^k corresponds therefore the infinitesimal transformation $e^j X_j f$ or $e^\mu X^\mu f$ in x^k, where

$$X_j f = \xi_j^k \, (x) \frac{\partial f}{\partial x^k} \quad ; \quad X_\mu f = \psi_\mu^j \, \xi_j^k \, (x) \frac{\partial f}{\partial x^k}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If $\underset{1}{v^\nu}$ and $\underset{2}{v^\nu}$ are two vectors which are (—)-constant and correspond therefore with two infinitesimal transformations

$$\delta^- \underset{1}{v^\nu} = 0 \quad ; \quad d\underset{1}{v^\nu} = -\overline{\Gamma}^\nu_{\lambda\mu} \, \underset{1}{v^\lambda} \, d\xi^\mu \, ; \, d\underset{1}{v^k} = 0,$$

$$\delta^- \underset{2}{v^\nu} = 0 \quad ; \quad d\underset{2}{v^\nu} = -\overline{\Gamma}^\nu_{\lambda\mu} \, \underset{2}{v^\lambda} \, d\xi^\mu \, ; \, d\underset{2}{v^k} = 0,$$

then we have

$$\underset{1}{df} = \underset{1}{v^j} \, X_j f \quad ; \quad \underset{2}{df} = \underset{2}{v^j} \, X_j f.$$

Now it is well known that

$$\underset{1}{v^i} \, \underset{2}{v^j} \, X_i \, X_j f - \underset{1}{v^i} \, \underset{2}{v^j} \, X_j \, X_i \, f = \underset{1}{v^i} \, \underset{2}{v^j} \, c_{ij}^{\cdot\cdot k} \, X_k f \quad . \quad . \quad . \quad (9)$$

where $c_{ij}^{\cdot\cdot k} = -c_{ji}^{\cdot\cdot k}$ are the $^1/_2 r^2 (r-1)$ constants of LIE, which determine the structure of the group. The corresponding linear element is the difference of

$$\underset{2}{v^\nu} + \underset{1}{v^\nu} + d\underset{2}{v^\nu}\,\underset{1}{} = \underset{2}{v^\nu} + \underset{1}{v^\nu} - \overline{\Gamma}^\nu_{\lambda\mu} \, \underset{1}{v^\lambda} \, \underset{2}{v^\mu}$$

and

$$\underset{1}{v^\nu} + \underset{2}{v^\nu} + d\underset{1}{v^\nu}\,\underset{2}{} = \underset{1}{v^\nu} + \underset{2}{v^\nu} - \overline{\Gamma}^\nu_{\lambda\mu} \, \underset{2}{v^\lambda} \, \underset{1}{v^\mu}$$

so that

$$\underset{1}{v^i} \, \underset{2}{v^j} \, c_{ij}^{\cdot\cdot k} = -\underset{1}{v^\lambda} \, \underset{2}{v^\mu} \left(\overline{\Gamma}^\nu_{\lambda\mu} - \overline{\Gamma}^\nu_{\lambda\mu}\right) \psi_\nu^k$$

or

$$S_{\lambda\mu}^{\cdot\cdot\nu} = \overline{\Gamma}^\nu_{[\lambda\mu]} = -{}^1/_2 \, c_{\lambda\mu}^{\cdot\cdot\nu} = -{}^1/_2 A_{\lambda\mu k}^{i\,j\,\nu} \, c_{ij}^{\cdot\cdot k} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

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[1]) Comp. f.i. LIE-SCHEFFERS, p. 376. We have used ξ and α instead of A_λ^j and A_i^λ to make comparison more easy.

$S_{\lambda\mu}^{\cdot\cdot\nu}$ is the quantity which makes the connexion asymmetrical. The connexion (—) is defined by the equation

$$\delta v^\nu = dv^\nu + \overset{0}{\Gamma}{}_{\lambda\mu}^{\nu}\, v^\lambda\, d\xi^\mu + S_{\lambda\mu}^{\cdot\cdot\nu}\, v^\lambda\, d\xi^\mu \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The $c_{ij}^{\cdot\cdot k}$ being constants, $S_{\lambda\mu}^{\cdot\cdot\nu}$ is a (—)-constant quantity

$$\overline{\nabla}_\omega S_{\lambda\mu}^{\cdot\cdot\nu} = 0 \;\; ; \;\; \delta^- S_{ij}^{\cdot\cdot k} = dS_{ij}^{\cdot\cdot k} = 0 \quad . \quad . \quad . \quad . \quad (12)$$

$S_{\lambda\mu}^{\cdot\cdot\nu}$ is only then equal to zero, if all the expressions $(X_i X_j)$ vanish.

§ 2. *The second connexion with zero curvature* (+).

A second connexion may be defined by means of the equation

$$T_\xi^{-1}\, T_{\xi+d\xi} = T_\eta^{-1}\, T_{\eta+d\eta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This connexion being equally integrable, the curvature is here zero as well. A geodesic passing through $\overset{0}{\xi}{}^\nu$ is given by the equation

$$T_\xi = T_{\xi_0}\, T_u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and is at the same time a geodesic of the (—)-connexion, since

$$T_{\xi_0}\, T_u = \left(T_{\xi_0}\, T_u\, T_{\xi_0}^{-1}\right) T_{\xi_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and $T_{\xi_0}\, T_u\, T_{\xi_0}^{-1}$ represents also a group with one parameter. Writing (13) in a finite form

$$T_\xi^{-1}\, T_{\xi'} = T_\eta^{-1}\, T_{\eta'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

it follows that the segments $\xi\,\xi'$ and $\eta\,\eta'$ are (+)-aequipollent. We will deduce this second connexion in another way and will find a formula corresponding with (11).

Introducing in the equation

$$'x = Tx$$

new variables on both sides,

$$'x' = S\,'x \quad ; \quad x' = S\,x$$

we have

$$'x' = S\,T\,S^{-1}\,x'.$$

S and T being transformations of the group, applying the transformation T on x is therefore the same as applying STS^{-1} on Sx. If we take S and T both infinitesimal

$$Sx^k = x^k + s^j\, X_j\, x^k\, dt + \ldots$$

$$Tx^k = x^k + t^j\, X_j\, x^k\, dt + \ldots$$

in which s^j and t^j are (—)-constant vectors, the transition from T into STS^{-1} corresponds with the linear element $s^j dt$ or, because

$$S^{-1} = x^k - s^j X_j x^k dt + \dots$$

$$TS^{-1} x^k = x^k - s^j X_j x^k dt + t^j X_j x^k dt - t^i X_i s^j X_j x^k dt^2 + \dots$$

$$STS^{-1} x^k = x^k + t^j X_j x^k dt - t^i s^j (X_i X_j - X_j X_i) x^k dt^2 + \dots$$

with the transition from t^k into $t^k + s^i t^j c_{ij}^{\cdot\cdot k} dt$. So we obtain a new connexion

$$\delta^+ t^k = - s^i t^j c_{ij}^{\cdot\cdot k} dt = t^i s^j c_{ij}^{\cdot\cdot k} dt = \delta^- t^k + t^i s^j c_{ij}^{\cdot\cdot k} dt \quad . \quad (17)$$

or in a more general form

$$\left.\begin{aligned} \delta^+ v^\nu &= \delta^- v^\nu + c_{\lambda\mu}^{\cdot\cdot\nu} v^\lambda d\xi^\mu \\ &= dv^\nu + \overset{0}{\Gamma}{}_{\lambda\mu}^{\nu} v^\lambda d\xi^\mu + {}^1/_2 c_{\lambda\mu}^{\cdot\cdot\nu} v^\lambda d\xi^\mu \\ &= dv^\nu + \overset{0}{\Gamma}{}_{\lambda\mu}^{\nu} v^\lambda d\xi^\mu - S_{\lambda\mu}^{\cdot\cdot\nu} v^\lambda d\xi^\mu \end{aligned}\right\} \quad . \quad . \quad . \quad (18)$$

By the (+)-connexion the infinitesimal transformations of the group undergo a transformation and these transformations form the *adjoint group*. Both connexions correspond therefore exactly with the two fundamental relations between the infinitesimal transformations which occur in the classical theory [1]).

The quantity $S_{\lambda\mu}^{\cdot\cdot\nu}$ is constant by the (+)-connexion also. Indeed we have

$$\left.\begin{aligned} \overset{+}{\nabla}_\omega S_{\lambda\mu}^{\cdot\cdot\nu} &= \overline{\nabla}_\omega S_{\lambda\mu}^{\cdot\cdot\nu} - c_{\lambda\omega}^{\cdot\cdot\alpha} S_{\alpha\mu}^{\cdot\cdot\nu} - c_{\mu\omega}^{\cdot\cdot\alpha} S_{\lambda\alpha}^{\cdot\cdot\nu} + c_{\alpha\omega}^{\cdot\cdot\nu} S_{\lambda\mu}^{\cdot\cdot\alpha} \\ &= 0 + \tfrac{2}{3} c_{[\omega\lambda}^{\cdot\cdot\alpha} c_{\mu]\alpha}^{\cdot\cdot\nu} = 0 \end{aligned}\right\} \quad . \quad . \quad (19)$$

since the JACOBIAN identity gives:

$$c_{[ij}^{\cdot\cdot m} c_{l]m}^{\cdot\cdot k} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The integrability of the (+)-connexion follows also by applying the general formula for the transformation of the quantity of curvature:

$$'R_{\omega\mu\lambda}^{\cdots\nu} = R_{\omega\mu\lambda}^{\cdots\nu} - 2\nabla_{[\omega} A_{|\lambda|\mu]}^{\cdot\cdot\nu} + 2\Gamma_{[\omega\mu]}^{\varkappa} A_{\lambda\varkappa}^{\cdot\cdot\nu} - 2 A_{\alpha[\omega}^{\cdot\cdot\nu} A_{|\lambda|\mu]}^{\cdot\cdot\alpha} \quad . \quad (21)$$

valid by the transformation of parameters

$$'\Gamma_{\lambda\mu}^{\nu} = \Gamma_{\lambda\mu}^{\nu} + A_{\lambda\mu}^{\cdot\cdot\nu}.$$

§ 3. *The symmetrical displacement* (0).

$S_{\lambda\mu}^{\cdot\cdot\nu}$ being an affinor in X_r, the equation

$$\delta^0 v^\nu = dv^\nu + \overset{0}{\Gamma}{}_{\lambda\mu}^{\nu} v^\lambda d\xi^\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

defines a third displacement (0), which is symmetrical.

$S_{\lambda\mu}^{\cdot\cdot\nu}$ is also constant by (0):

$$\overset{0}{\nabla}_\omega S_{\lambda\mu}^{\cdot\cdot\nu} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

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[1]) The two expressions (2) and (13) occur for the first time side by side in CARTANs paper Bull. d. Sc. Math. 34. 1910, page 250—283.

in consequence of the JACOBIAN identity. By applying (21), we have for the quantity of curvature

$$\left.\begin{aligned} \overset{0}{R}{}_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} &= -2\,S_{\omega\mu}^{\cdot\cdot\alpha}\,S_{\lambda\alpha}^{\cdot\cdot\nu} - 2\,S_{\alpha[\omega}^{\cdot\cdot\nu}\,S_{|\lambda|\mu]}^{\cdot\cdot\alpha} \\ &= -2\,S_{\omega\mu}^{\cdot\cdot\alpha}\,S_{\lambda\alpha}^{\cdot\cdot\nu} - S_{\lambda\mu}^{\cdot\cdot\alpha}\,S_{\alpha\omega}^{\cdot\cdot\nu} + S_{\lambda\omega}^{\cdot\cdot\alpha}\,S_{\alpha\mu}^{\cdot\cdot\nu} \\ &= S_{\omega\mu}^{\cdot\cdot\alpha}\,S_{\alpha\lambda}^{\cdot\cdot\nu} \end{aligned}\right\} \quad . \quad . \quad (25)$$

from which it follows that $\overset{0}{R}{}_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ is constant for all the three connexions.

The part of the parameters that is symmetrical in $\lambda\mu$ being equal for the three connexions, it follows that they have the same geodesics. The geometrical link between the three connexions is as follows. If $d\xi^\nu$ and $\overset{1}{d}\xi^\nu$ are two linear elements in ξ^ν, and if $\overset{0}{d}\xi^\nu$, $\overset{+}{d}\xi^\nu$ and $\overset{-}{d}\xi^\nu$ arise by (0)-, (+)- and (−)-displacement of $d\xi^\nu$ along $\overset{1}{d}\xi^\nu$ and $\overset{0}{d}\xi^\nu$, $\overset{+}{d}\xi^\nu$ and $\overset{-}{d}\xi^\nu$ by the analogous displacement of $\overset{1}{d}\xi^\nu$ along $d\xi^\nu$, then $\overset{0}{d}\xi^\nu$ and $\overset{0}{d}\xi^\nu$ form together with $d\xi^\nu$ and $\overset{1}{d}\xi^\nu$ a closed figure, as well as $\overset{+}{d}\xi^\nu$ and $\overset{-}{d}\xi^\nu$ and $\overset{-}{d}\xi^\nu$ and $\overset{+}{d}\xi^\nu$, while among the three points opposite to ξ^ν, the point belonging to $\overset{0}{d}\xi^\nu$ and $\overset{0}{d}\xi^\nu$ lies in the midst between the other two. If $f^{\lambda\mu}\,d\sigma$ is an infinitesimal bivector, and if we call generally $f^{\lambda\mu}\,d\sigma\,\Gamma_{[\lambda\mu]}^{\nu}$ the corresponding torsion, it follows that the (+)-torsion and the (−)-torsion are equal and of opposite sign, the (0)-torsion being zero. If the bivector is formed by the two infinitesimal transformations X_i and X_k, then the torsion has the direction of $(X_i\,X_k)$.

§ 4. *The adjoint group.*

If we write always e^k for a (−)-constant vector, which corresponds therefore with an infinitesimal transformation $e^k\,X_k$, the (+)-differential is according to (18) given by the equation

$$\delta^+ e^k = e^i\,c_{ij}^{\cdot\cdot k}\,d\xi^j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Any function f of e^k undergoes by an infinitesimal transformation of the adjoint group a change

$$df = \frac{\partial f}{\partial e^k}\,e^i\,c_{ij}^{\cdot\cdot k}\,d\xi^j\,,$$

so that we get for the symbol of LIE of the infinitesimal transformations the well known equation

$$E_j\,f = e^i\,c_{ij}^{\cdot\cdot k}\,\frac{\partial f}{\partial e^k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If we call *constant quantities* such quantities, which are constant by (+), (−) and (0), then the following propositions hold good:

A constant simple co- or contravariant p-vector determines a $(n-p)$-fold, resp. p-fold invariant subgroup of the adjoint group.

The system of equations, obtained by combining in an invariant way a constant quantity with one or more vectors e^k and putting this combination equal to zero, is invariant by the transformations of the adjoint group. If the system is linear in e^k, it defines an invariant subgroup of the adjoint group, which may be however zero.

Examples:

$$v_{i_1 \ldots i_p}^{\ldots k_1 \ldots k_q} \; e^{i_1} \ldots e^{i_s} = 0 \quad ; \quad s \leqq p$$

$$v_{i_1 \ldots i_p}^{\ldots [k_1 [k_2 \mid k_3 \ldots k_q \mid} e^{h]} e^{l_2]} = 0.$$

Constant quantities are among others $c_{ij}^{\cdot\cdot k}$ and $\overset{0}{R}{}_{lij}^{\cdot\cdot\cdot k}$ and all their covariants. The most important ones are

$$\overset{0}{R}_{ij} = \overset{0}{R}{}_{lij}^{\cdot\cdot\cdot l} \quad ; \quad R_{[ij]} \quad ; \quad R_{(ij)} \tag{28}$$

$$g_i = {}^1/_2 \, c_{ia}^{\cdot\cdot a}$$

$$c \, g_{ij} = {}^1/_4 \, c_{ai}^{\cdot\cdot b} \, c_{bj}^{\cdot\cdot a} ; \; c = \text{constant} \tag{29}$$

$$g_{ijk} = - {}^1/_8 \, c_{ai}^{\cdot\cdot b} \, c_{bj}^{\cdot\cdot c} \, c_{ck}^{\cdot\cdot a} \; ^1)$$

All quantities (29) admit cyclical permutation of the suffixes:

$$g_{ijk} = g_{jki} = g_{kij} .$$

Important invariant subgroups are:

a. the group of g_i

$$e^j g_j = 0.$$

b. the central group, containing all infinitesimal transformations whose alternated combinations (Klammerausdruck, crochet) with every other vanish

$$e^i c_{ij}^{\cdot\cdot k} = 0.$$

c. the derived group, containing all infinitesimal transformations, which may be written as alternated combination of two others

$$c_{ij}^{\cdot\cdot [k} e^{l]} = 0.$$

d. the group of g_{ij}

$$e^i g_{ij} = 0.$$

The tensor g_{ij} has remarkable properties. In the first place it follows from (25)

$$R_{ij} = c \, g_{ij} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{30}$$

R_{ij} is therefore symmetrical, i. c. the connexion (0) is equiaffine (inhaltstreu). The connexions (+) and (−) are integrable and have therefore the same property. HURWITZ [2]) has used the equiaffinity of (−) to

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[1]) The constants c, $^1/_4$, etc. are introduced with regard to calculations to be executed further on.

[2]) Göttinger Nachrichten 1897 p. 71—90.

form volume-integrals in the group-manifold of an arbitrary finite continuous group. If we start from an unit of volume in ξ^ν and denote by $d\tau^0$, $d\tau^+$ and $d\tau^-$ the volumes of a space-element, measured with these unit, then generally $d\tau^0$, $d\tau^+$ and $d\tau^-$ are unequal. Moreover, $d\tau^+ = d\tau^-$ in the E_{r-1} of the invariant subgroup of g_i passing by ξ^ν, whilst always $d\tau^0 = \sqrt{d\tau^+ d\tau^-}$. Only in the case $g_i = 0$ the measured volumina get equal in every point.

Applying the JACOBIAN identity, we have

$$\left.\begin{aligned} S_{ij}^{\cdot\cdot l}\, g_{lk} &= {}^1\!/_c\, S_{ij}^{\cdot\cdot l}\, S_{lm}^{\cdot\cdot n}\, S_{kn}^{\cdot\cdot m} \\ &= -\frac{1}{c} S_{jm}^{\cdot\cdot l}\, S_{li}^{\cdot\cdot n}\, S_{kn}^{\cdot\cdot m} \\ &\quad -\frac{1}{c} S_{mi}^{\cdot\cdot l}\, S_{ij}^{\cdot\cdot n}\, S_{kn}^{\cdot\cdot m} = \\ &= -\frac{1}{c} g_{jik} + \frac{1}{c} g_{ijk} = -\frac{1}{c} g_{ikj} + \frac{1}{c} g_{ijk} \end{aligned}\right\} \quad . \quad . \quad (31)$$

whence it follows that $S_{ij}^{\cdot\cdot l}\, g_{lk}$ is alternating in jk and therefore is a trivector.

§ 5. *Semi-simple and simple groups.*

A simple group is a group, which does not possess invariant subgroups. A group is called semi-simple, if it possesses no integrable invariant subgroups. CARTAN has proved [1]) that a group is then and only then semi-simple, if the tensor $g_{\lambda\mu}$ has the rank r. There exists therefore in that case a tensor of rank r, which is (—)-constant and it follows that for a semi-simple group the connexion (0) is a Riemannian one with $g_{\lambda\mu}$ as fundamental tensor [2]). For the constant c it follows from (30):

$$c = \frac{1}{n} K = -(n-1) K_0 \quad ; \quad K = K_{\alpha\beta}^{\cdot\cdot\beta\alpha}; \quad K_{\omega\omega\lambda}^{\cdot\cdot\cdot\nu} = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} \quad . \quad . \quad (32)$$

Moreover it follows that g_λ is zero.

If the group is semi-simple, there exist invariant sub-groups. Such a subgroup determines a real p-direction which is constant by the three displacements [3]). These p-directions determine a system of ∞^{n-p} V_p which are totally geodesic and mutually parallel. The $(n-p)$-direction orthogonal to these V_p is also constant and determines in the same way ∞^p totally geodesic mutually parallel V_{n-p}. The coordinates may be

[1]) Thèse, Paris 1894.

[2]) L. P. EISENHART, Proc. Nat. Acad. 11, 1925, p. 246—250, has proved that every simple transitive group is connected with a symmetrical connexion and with one or more symmetrical quantities of degree 2.

[3]) Compare the division of semi-simple groups by CARTAN, Thèse, p. 52, especially for the demonstration that the case where the direction would have in common a p-direction with the orthogonal $(n-p)$-direction, can not occur.

chosen in such a way that their congruences are entirely in the V_p and V_{n-p}, so that the linear element decomposes into

$$ds^2 = \underset{1}{ds^2} + \underset{2}{ds^2}$$

where $\underset{1}{ds}$ represents the linear element of V_p and $\underset{2}{ds}$ the linear element of V_{n-p}. $S_{\lambda\mu\nu}$ and $R_{\omega\mu\lambda\nu}$ decompose into two separately constant parts which lie totally in V_p resp. V_{n-p}. If there are more invariant subgroups, then it is possible to repeat this decomposition, so that V_r finally decomposes into a certain number of V_{p1}, V_{p2} etc., which are all geodesic, parallel and mutually orthogonal. Every system belongs to a simple group, which is an invariant subgroup of the given group, and every V_p of this system is groupmanifold of this subgroup. The geometry of the V_r decomposes into the geometries of these simple invariant subgroups. The geometry of a semi-simple group is therefore reduced to the geometries of simple groups. [1])

§ 6. *Some geometrical properties.*

Starting from an arbitrary geodesic we obtain by means of the connexion (+) a congruence of geodesics which are mutually (+)-parallel, and in the same manner we obtain a congruence of mutually (—)-parallel geodesics. For $r = 3$ it follows from (30) that V_3 is an S_3. The two parallelisms coincide then with the parallelisms of CLIFFORD. [2])

If v^ν is a tangential vector in a point of the first geodesic, then the first congruence is given by the equation

$$\overset{+}{\nabla}_\mu v^\nu = \overset{0}{\nabla}_\mu v^\nu + S_{\lambda\mu}^{\cdots\nu} v^\lambda = 0,$$

from which it follows that v_λ is a solution of the equation of KILLING

$$\nabla_{(\mu} v_{\lambda)} = 0. \text{ [3])}$$

The transformation

$$'\xi^\nu = \xi^\nu + v^\nu \, dt$$

is therefore a *motion*, and even a *translation* since the length of the vector v^ν is the same in all points of V_r. The same consideration being

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[1]) H. LEVY, Rend. Acc. Linc. 3, 1926, p. 65—69 (see also p. 124—129) has found that every Riemannian geometry with $\nabla_\xi K_{\omega\mu\lambda\upsilon} = 0$ decomposes into Riemannian geometries with zero curvature. This does not agree with our results.

[2]) A treatment of the geometry in S_3 and of CLIFFORD's parallelisms from the point of view of the different connexions is to be found by CARTAN, Récentes généralisations de la notion d'espace, Bull. Sc.. Math. 48, 1924, p. 294—320 and from another point of view by EN. BORTOLOTTI, Parallelismo assoluto e vincolato negli S_3 etc. Atti Veneto 84, 1925, p. 821—858.

[3]) Der Ricci Kalkül, SPRINGER 1924, p. 212.

valid for the displacement (—), two kinds of translations are possible in V_r .

The first kind may be given also by the equation

$$T_{'\xi} = T_a \, T_\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where T_a represents a definite transformation, the second by

$$T_{'\xi} = T_\xi \, T_a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

whence it appears that the translations belong to the first and second parametergroup. The described paths are (—)-aequipollent, while a segment passes into a (+)-aequipollent segment. The same holds m.m. for the second transformation, and for all continuous groups, not only for the semi-simple ones. Transformations of the form

$$T_{'\xi} = T_a \, T_\xi \, T_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

leave invariant all three displacements, while (+)- as good as (—)-parallel segments pass into segments of the same kind. We may ask if there exist yet other pointtransformations of X_r , which leave invariant the connexions (+) and (—). Determining first the transformations wich leave invariant the group-structure, i.e. which transform $T_\xi \; T_\eta = T_\zeta$ into $T_{'\xi} \; T_{'\eta} = T_{'\zeta}$ [1]), to these transformations only the transformations (33) or (34) have to be added to obtain the desired transformations. The transformations (35) form an invariant subgroup of the so obtained group.

The connexions (+) and (—) determine each the structure of the group entirely. On the contrary, the group being not simple or semi-simple, it may occur that the structure is not entirely determined by (0). In the first place the group of transformations of (0) in itself contains among others the transformations $T_{'\xi} = T_\xi^{-1}$ that changes (+) in (—) v.v. The group being not simple or semi-simple there may exist moreover transformation of (0) in itself, conserving not the total of (+) and (—) parallelism. If this case occurs, there exist different absolute parallelisms. A simple example is given by the group

$$(X_1 \, X_2) = X_3 \quad ; \quad (X_1 \, X_3) = (X_2 \, X_3) = 0.$$

The (0)-connexion is here an ordinary affine one. In the corresponding E_3 besides the ordinary parallelisms there can be defined an infinity of absolute parallelisms with the straight lines as geodesics. With the given group is connected the parallelism by which the direction (α, β, γ) in 0 corresponds with the direction $(\alpha, \beta, \gamma + \alpha\, y - \beta\, x)$ in (x, y, z).

We can indicate also in which case a given integrable connexion is

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[1]) For the semi-simple groups this investigation has been executed by CARTAN, Bull. des Sc. Math. 49, 1925, p. 361—374.

(—)- or (+)-connexion of a group. If $A\,B\,C$ is a triangle of geodesics in X_r and if the connexion belongs to a group, then $A'\,C'$ is aequipollent with $A\,C$, if this property holds good for $A'\,B'$ and $A\,B$ as well as for $B'\,C'$ and $B\,C$, for every choice of A'.

The reciprocal is also true. If we make correspond with each segment $\overrightarrow{a}$ in X_r and its aequipollent segments the transformation, which transforms an arbitrary point X into $'X$, so that $\overrightarrow{X\,'X} = \overrightarrow{a}$, then these transformations form a continuous group with r parameters and the given connexion is the (—)-connexion of this group. Infinitesimally the condition is as follows. The connexion belongs then and only then to a group, if the expression $f^{\lambda\mu}d\sigma\, S_{\lambda\mu}^{\cdot\cdot\,\nu}$ does not change if the surface element is replaced by an aequipollent element, or, which is the same, if $S_{\lambda\mu}^{\cdot\cdot\,\nu}$ is a quantity constant by the given connexion,

Starting from two vectors v^ν and w^ν in ξ^ν, we may construct a congruence v^ν by means of the connexion (+) and a congruence w^ν by means of the connexion (—). These two congruences are V_2-building [1]), since

$$v^\mu \overset{0}{\nabla}_\mu w^\nu - w^\mu \overset{0}{\nabla}_\mu v^\nu = v^\mu S_{\lambda\mu}^{\cdot\cdot\,\nu} w^\lambda + w^\mu S_{\lambda\mu}^{\cdot\cdot\,\nu} v^\lambda = 0.$$

In any of these V_2 the angle between two vectors v^ν and w^ν in every point is equal, since

$$v^\mu \overset{0}{\nabla}_\mu v_\lambda\, w^\lambda = v^\mu\, S_{\lambda\mu}^{\cdot\cdot\,\nu} v_\nu\, w^\lambda + v^\mu\, S_{\mu\lambda}^{\cdot\cdot\,\nu} w^\lambda\, v_\nu = 0$$

$$w^\mu \overset{0}{\nabla}_\mu v_\lambda\, w^\lambda = 0.$$

In the V_2 obtained in this manner two translations are possible whence it follows that their geometry must be euclidian. For $r=3$ the V_2 are the well known surfaces of CLIFFORD.

We may generalise these considerations. If T_u is the general transformation of a p-fold continuous group, $T_u\,T_\xi$ is a total geodesic X_p passing through ξ^ν. Comparing this X_p with the X_p passing through η^ν given by $T_u\,T_\eta$, with every geodesic in one X_p corresponds a (—)-parallel in the other. Indeed, T_u and T_v being two general transformations of the subgroup, the segment $T_v\,T_\xi\,(T_u\,T_\xi)^{-1}$ is (—)-aequipollent with the segment $T_v\,T_\eta\,(T_u\,T_\eta)^{-1}$ since

$$T_v\,T_\xi\,T_\xi^{-1}\,T_u^{-1} = T_v\,T_u^{-1} = T_v\,T_\eta\,T_\eta^{-1}\,T_u^{-1}.$$

We may therefore say that X_p displaces itself (—)-parallel if ξ^ν changes. Choosing η^ν in the first X_p it follows that, being given a geodesic in that X_p passing through ξ^ν, through every other point of that X_p passes a (—)-parallel geodesic, situated entirely in X_p. In the same way $T_\xi\,T_\nu$

[1]) Ricci Kalkül, p. 107.

gives a totally geodesic X_p through ξ^ν, for which the same holds good with respect to the displacement (+). These two X_p coincide if T_ξ is a not singular transformation of the subgroup, since

$$T_\xi T_u = (T_\xi T_u T_\xi^{-1}) T_\xi$$

and $T_\xi T_u T_\xi^{-1}$ represents a general transformation of the subgroup also. The property of parallel motion of geodesics exists also for the (+)-parallelisms.

The reciprocal holds good: if it is possible to make correspond to every geodesic X_p through every point a (—)- (or (+)-) parallel, which lies entirely in X_p, then X_p belongs to a subgroup.

The X_p arising from the X_p through ξ^ν by parallel motion are generally different, according as the (+)- or (—)-displacement is used.

The invariant subgroups distinguish themselves by the coincidence of the two systems of X_p, which are obtained from the X_p in ξ^ν by means of (+) and (—).

In another kind of totally geodesic X_p the (+)- or (—)-parallel of a geodesic through a point of the X_p not situated on that line lies not in general in X_p. (A (o)-parallel direction is always situated in X_p). If we construct through a point not situated in X_p all geodesics, which are (+)- or (—)-parallel with all geodesics through a definite point of X_p, then there arises a totally geodesic X_p, but this X_p is different according as the point in the first X_p is chosen differently. In an S_3 exist only real geodesical S_2 of the second kind, since the corresponding group of the ordinary rotations in R_3 contains no subgroups with two parameters. The isotropical planes belong however to the first kind.

§ 7. *Classification of the geometries belonging to the simple groups.*

For a classification of the geometries of simple groups we have to start with the classification of these groups given by CARTAN [1]). By this classification a great part is played by a number l that has the following geometrical signification. If v^ν is a definite vector, which has not a singular direction, the vectors w^ν, satisfying the equation

$$v^\omega w^\mu R_{\omega\mu\lambda\nu} = 0$$

fill an R_l. There exist four normal types, A, B, C and D:

A.	$l = 1, 2, \ldots$	$r = l(l+2)$	$= 3, 8, 15, 24, 35$
B.	$l = 3, 4, \ldots$	$r = l(2l+1)$	$= 21, 36, 55, 78, 105$
C.	$l = 2, 3, \ldots$	$r = l(2l+1)$	$= 10, 21, 36$
D.	$l = 4, 5, \ldots$	$r = l(2l-1)$	$= 28, 45, 66, 91, 120$

and five abnormal ones:

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[1]) Thèse 1894.

$E.$	$l=6$	$r=78$
$E.$	$l=7$	$r=133$
$E.$	$l=8$	$r=248$
$F.$	$l=4$	$r=52$
$G.$	$l=2$	$r=14.$

For each type the form of $S_{\lambda\mu\nu}$ may be deduced and the corresponding geometry constructed [1]).

We will draw the attention on some important questions. Realising V_r in R_{r+m} it would be important to know the minimum value of m and to have a method to obtain such a realisation in the most simple manner. Also there may be asked for the necessary and sufficient conditions for a given symmetrical connexion being (o)-connexion of a group.

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[1]) Compare WEYL. Math. Zeitschr. 24, 1925 p. 354 f.

Physiology. *"On the reaction course of physiological buffermixtures, examined by direct registration of the P_H changes"*. By Prof. F. J. J. BUYTENDIJK and R. BRINKMAN. (Communicated by Prof. H. ZWAARDEMAKER).

(Communicated at the meeting of February 27, 1926).

In the field of study of the biological neutrality regulation, in spite of the steady increase in detail, the most important problem is still the properties of the buffer-system as formed by the carbonic acid and its salts, the phosphates, the oxy-haemoglobin and perhaps by other weak acids. These systems have been so exhaustively treated in the well-known researches of L. HENDERSON, SPIRO, SÖRENSEN, HASSELBACH, MICHAELIS, WARBURG, VAN SLYKE and many others, and their study has given us such a deep insight into the mechanism of the biological regulation of the acidity, that we have an impression of the comparative completeness of our knowledge. The reaction of the respiratory centrum, of the function of the kidneys and other P_H regulating organs on the change in concentration of the buffer components may still be frequently less clear, but there is evidently little to add to our physico-chemical knowledge of the central buffer-systems, at least of the carbonic acid-bicarbonate and of the phosphate system.

The aim of this communication is to draw attention to the changes which *occur in a bicarbonate and phosphate system before an equilibrium is attained.*

It was obvious that the chemical examination of the neutrality regulation, as a typical application of the law of mass-action with ion-reactions, showed only the equilibrium conditions but not the course of the reaction. And yet there is, e.g. in the properties of the carbonic acid certainly a reason to examine this course of reaction more in detail, since the non-dissociated H_2CO_3, according to THIEL and STROHECKER, may be present for a short time after its formation as the strong oxy-formic acid, and in that time it may penetrate into cells by its very strong solubility and permeability. Thus for these reasons, besides by their difference in buffering action, the systems

$$\frac{[H_2CO_3]}{[Na\,HCO_3]} \text{ and } \frac{2\,[H_2CO_3]}{2\,[Na\,HCO]_3}$$

are by no means equal in their biological effect.

Thus, although the system $HCO'_3 + H \rightarrow H_2CO_3 \rightleftarrows CO_2 + H_2O$ or perhaps $HCO'_3 \leftrightarrows CO_2 + OH'$ is exactly known in its state of equilibrium, the time required for the accomplishment of this equilibrium is so long

that, for instance, the strong momentary H_2CO_3 might have a great biological influence. Of the duration of its existence, however, very little is known.

On the other hand, COLLINGWOOD[1]) has pointed out the phenomenon that when a current of gaseous CO_2 be passed 2—3 seconds through a "weak alkaline" solution and all the CO_2 above the solution carefully blown off, it may take as long as 30 seconds before the addition of phenol red as indicator will show a change to a more acid reaction. He concluded that if CO_2, formed in the tissues, were to be transported by the blood as HCO'_3 the formation time for this ion to be produced ($CO_2 + OH' = HCO'_3$) is much to long to ensure a sufficiently rapid transport of CO_2, so that there must be still unknown factors in this mechanism.

An indication that in the formation of the bicarbonate-carbonic acid equilibrium intermediate reactions may occur, whereby the time factor biologically becomes important in this equilibrium, is found in a simple experiment, part of which is described by MICHAELIS: Add to a solution of 0.1 N $NaHCO_3$ a little neutral red, which is yellow in alkali-sol.; add enough 0.1 N HCl to turn the colour red after mixing. *After a few seconds* the colour will change back to yellow; if more HCl be added the change to yellow-red-yellow will ensue.

Loss of CO_2 is not the cause of this phenomenon which, according to THIEL and STROHECKER, depends upon the relatively smaller velocity of the reaction $H_2CO_3 \rightarrow H_2O + CO_2$.

There is, however, a similar experiment of which we find no mention. If we take again 0.1 N $NaHCO_3$, add to it phenol red as indicator and then a little 0.1 N HCl, *the colour change from red to yellow will begin a few seconds after the mixing*, and not suddenly but gradually. And this colour does not change back to the alkaline red tint.

The field of change of phenol red lies between P_H 8.4—7.4 of neutral red between P_H 8.7.

From this simple experiment it may already be assumed that the increase of the hydrogen ion concentration in a bicarbonate sol. to which acid has been added, would, in the vicinity of the neutral reaction, not take place so regularly as might be expected. That the difference in the manner of change is not a peculiarity of both indicators themselves is evident from the fact that in the phosphate solutions of analogous reaction, buffer capacity and molarity, the change takes place *in both always* immediately and definitely.

A further examination of the process of buffer reactions was only possible by a method which directly indicated the momentary [H] with as little latency as possible.

[1]) Proc. Physiol. Soc. July 5th, 1925, Journ. Physiol. 59, XXII, 1924—25.

For this, an ordinary electrometric P_H determination with compensation connections as zero-method is not practicable, but we must be able to follow direct the strenght of the E. M. F. produced between the solution examined and a standard sol., and by preference be able to register it. A galvano-meter method with a lead-off by common Pt. electrodes is not possible owing to polarisation phenomena.

TREADWELL and WEISS [1]) have indeed indicated an H electrode which is nearly non-polarisable and which can sustain a permanent very slight lead-off of current, but this requires a passage of gaseous hydrogen, is very complicated, and acts for our purpose too slowly.

A purely electro-metric method with a binant or quadrant electro-meter as measuring instrument likewise appeared less well adapted, owing to the instrument adjusting slowly, whereby, moreover, it is impossible to register a rapid change of the E. M. F. to be measured in two directions.

Very suitable, however, appeared to us a triode-lamp connection, whereby the E. M. F. as the voltage on the grid influences the discharge current between the filament and the plate, as has been indicated by GOODE [2]). This system is unpolarisable, and the variation of voltage anode involves a change of current which may be multiplied and registered.

Moreover, in order to trace directly the P_H to be measured, a hydrogen-electrode, which was in equilibrium with the free gas, could not be used as electrode, so we employed the homogenous system with a constant hydrogen pressure by the equilibrium chinon $+ H_2 =$ hydrochinon (so-called chinhydronelectrode), and in which the electrode adjusts itself almost instantaneously.

When using the beaker electrode, after MISLOWITZER [3]), in which the conducting and comparison electrodes are combined, blank platinum electrodes are used, and all connecting pieces are avoided. The electrometric determination of [H] is by this method very simple, while in the range of the H-ion concentration, which we will examine, the chinhydronelectrode yields results as exact as the classic H-electrode.

As galvanometer the string-galvanometer was the proper instrument, with which every variation in current could be instantaneously registrated.

Schematically our method was as follows:

The buffer-system to be examined was in the inmost beaker of the chinhydronelectrode. Very vigorous stirring with the Pt-electrode itself ensures as rapid as possible a mixing of the added acid. The E.M.F. to be measured is conducted through a potentiometer where it can be exactly compensated and measured. From this potentiometer the

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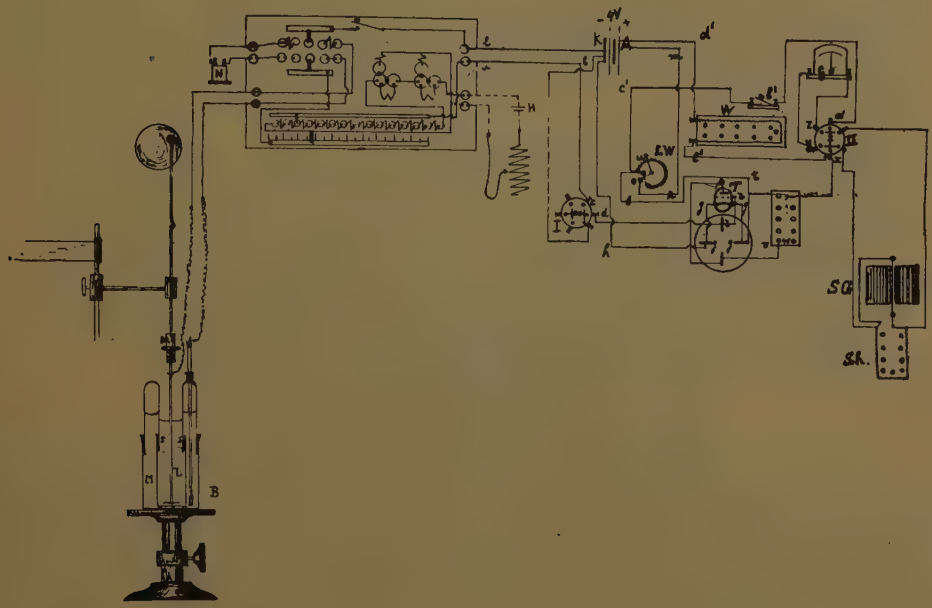
[1]) Helv. Chim. Acta 1, 410 (1919).
[2]) Journ. Amer. Chem. Soc. 1922.
[3]) Biochem. Zeitschr. **159** 58 (1925).

tension arrives at the grid of a triode-lamp, while the variation of the compensated anode-current of this lamp is registered with the string galvanometer.

For the exact description see diagram below:



B is a chinhydron electrode vessel model I, after MISLOWITZER [1]), in which a ground surface S wetted with saturated KCL forms the conductive connection between the standard buffer solution in the outermost beaker (M) and the unknown solution in the inner beaker (L). The outer solution (in M) is a mixture of equal parts of saturated KCL and a standard acetate solution (100 cc. N. $NaOH$ + 200 cc. N. CH_3COOH + + 700 aq. $P_H = 4.62$). Both liquids are saturated with chinhydron.

For a rapid registration of the effect of an acid addition on the P_H vigorous stirring is of great importance. We obtained the best results with a disk electrode measuring $^5/_6$ diameter of the inner beaker lumen and moving rapidly up and down, over the whole length of the fluid column with a frequency of about 10 per sec.

The E.M.F. to be measured can be compensated by potentiometer, drawn schematically in the figure above. The compensation was accurate up to $^1/_4$ millivolt. (N standard cell, H additional battery.)

The E.M.F. reaches through the circuit *a b c d r g h k l* the grid of the triode-lamp.

The lamp is a miniwatt lamp B 406, with a plate potential of 36 V. A small Weston needle galvanometer with a sensibility of 22 micro-amp. serves as zero-instrument. After compensation the discharge current could be directed by the POHL's reverser II to the stringgalvanometer SG (with shunt Sh). [2])

The sensitivity of this apparatus easily exceeds 0.02 P_H for a displacement of 1 mM of the string shadow, and might even be increased by using more than one triode lamp.

[1]) l. c.

[2]) Our thanks are due to Mr. HUIZING for his valued explanation.

In the first place now we examined how the P_H varied in the time period immediately after the addition of diluted HCl to a 0.02 N. solution of $NaHCO_3$ and Na_2HPO_4.

The effect of adding separately 1 c.c. 0.04 N. HCl to 40 c.c. buffer mixture (i. e. each time 0.001 N. HCl) proved to be as follows:

With the phosphate solution the calculated P_H was reached immediately, and this P_H remains constant. The speed of the adjustment is entirely dependent upon the speed of stirring, and the phenomenon remains the same in the entire P_H range examined (8.5—6.5). Only the buffer capacity is naturally greatest near the neutral reaction.

Fig. 1 gives an example.

Fig. 1. Addition of diluted HCl to 0.02 N. $Na_2HPO_4 + NaH_2PO_4$. P_H changes from 7.93 to 7.35.
1. Time in secs.
2. String shadow in which stirring is visible.
3. Signal for the acid to be added.

The process in a bicarbonate system was entirely different, as can be seen in Fig. 2.

The variation of P_H thus actually took place as the indicator reactions had led us to suppose; from P_H 8.3 to P_H 7.7 added acid is at once completely buffered, and not until some tenths of seconds later does the reaction reach its equilibrium value.

Between P_H 7.7—7.5 the rise becomes steeper, and with P_H 7.4 a typical „acid summit" is seen, which is very probably caused by the existence of H_2CO_3 during 1—2 seconds.

It is very interesting to note that this variation in the reaction should just occur in the range of blood reaction, where one may expect the one type as well as the other.

We see from this observation thus that the short duration of the strong H_2CO_3 in vivo must certainly be taken into account.

We cannot in this communication go further into this question, but

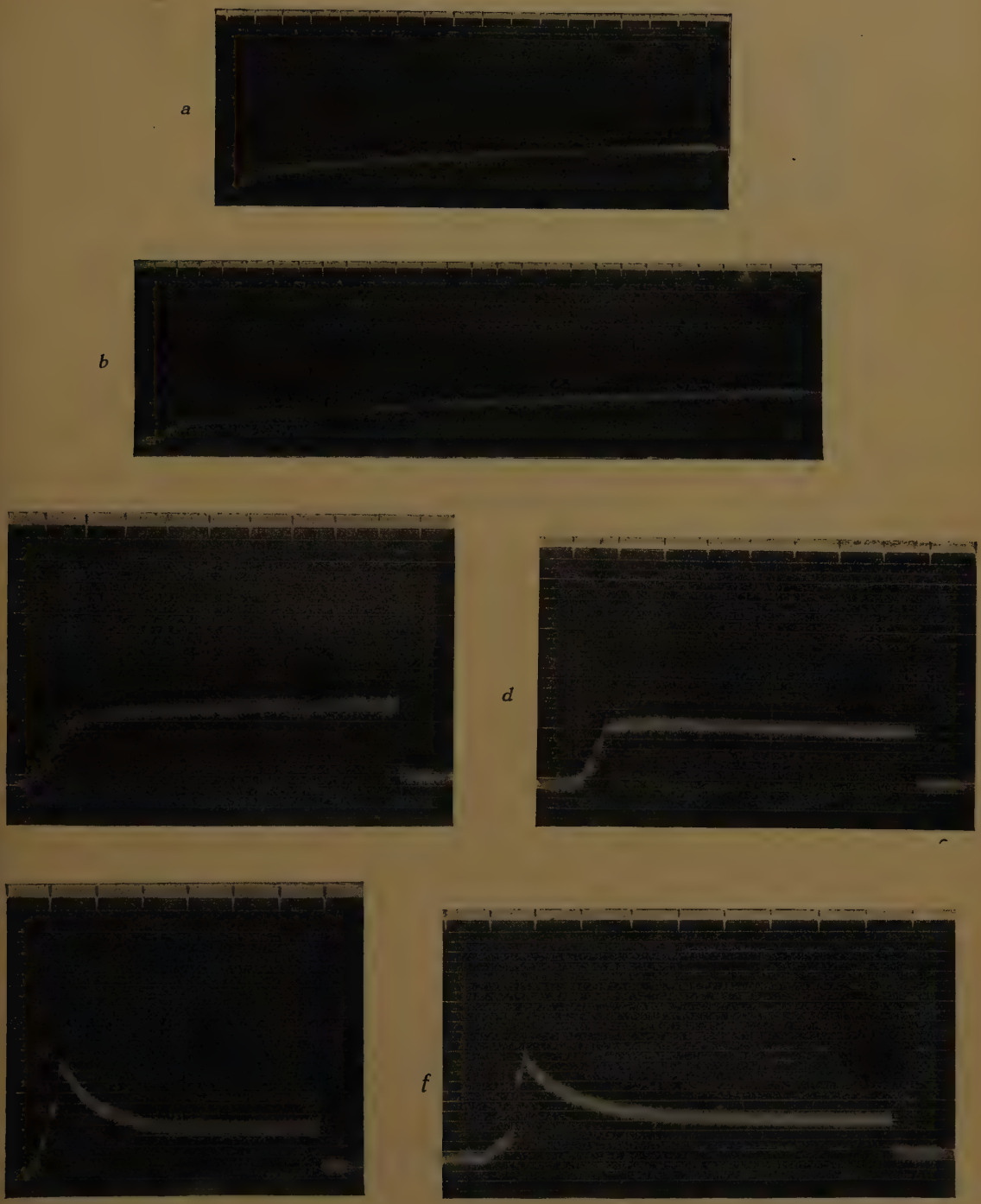


Fig. 2. Successive adding of 0.001 N. HCl to 0.02 N. $NaHCO_3$
Time in seconds.

d 0.004 N. HCl, P_H of 7.58—7.45
e 0.005 N. HCl, P_H of 7.42—7.36
f 0.006 N. HCl, P_H of 7.36—7.28
a 0.001 N. HCl, P_H of 8.31—8.03
b 0.002 N: HCl, P_H of 8.03—7.77
c 0.003 N. HCl, P_H of 7.77—7.58

wish to point out again that with a continued production of acid the H_2CO_3 may be permanently present in a larger concentration if the rapidity of the formation of this acid is at all comparable to the slowness of the CO_2 decomposition.

The actual P_H in an acidotic state can thus not be determined in the usual way, but must be examined in the living blood in the way we have indicated. Then it is possible that a more acid reaction may be found than has hitherto been observed.

Besides in the investigations as to the velocity of the buffer reactions, and in particular as to the significance of free H_2CO_3, we suppose that the method of registration above described may be useful in the study of different problems to which we hope to return later. We have here in view especially the direct registration of blood and tissue-fluid reactions in vivo, the measurement of biological membrane potentials and their change under nerve stimulation, and the direct registration of the carbonic acid percentage of the air exhaled.

Physiological Laboratorium of the State University.
Groningen, February 1926.

Botany. — *"The heredity of the need of light for germination in tobacco-seeds."* By J. A. HONING. (Communicated by Prof. A. H. BLAAUW.)

(Communicated at the meeting of March 27, 1926).

The researches of CIESLAR (1883), RACIBORSKI (1900), JENSEN (1901 and 1902) and KINZEL (1907) proved, that tobacco-seed needs light to germinate. GASZNER (1915) however counts *Nicotiana Tabacum* among the "Samen mit indifferenter Lichtwirkung" and found that the germination in darkness even occurred a little more rapidly. This made me surmise that in the great group of types, joined in the Linnean species *N. Tabacum*, there must exist considerable differences as to the need of light to germinate. In order to find this out I examined (1916) 51 samples of tobacco-seeds, viz. 8 from Deli, 6 from Western and Central Europe, 14 from the Balkan and Asia Minor, 21 from America, 1 of *N. quadrivalvis* and 1 of *N. rustica*, every time 300 seeds in darkness and 300 in diffuse day-light.

The samples of Deli-tobacco did not germinate at all or badly in the dark, in diffuse day-light very well, most of them for more than 90 % in 3 days. The seeds of types from South-Eastern Europe and Asia Minor however germinated excellently in darkness, for 50 to 94 % in 7 days, those of Western and Central Europe varied greatly, from 2 to 68 % and of the 21 American specimens 10 had not yet started after 7 days. This I considered a satisfactory explanation of the difference in results of CIESLAR a.o. as contrasted with GASZNER's.

I was rather sorry not to get acquainted with GOODSPEED's publication (1919) before 1925, whose experiences, were more in correspondence with GASZNER's and who repeated my experiments with *N. Tabacum* and *rustica* for this reason. GOODSPEED writes: "I am at a loss to explain this result of HONING or a number of the others which he reports." The explanation may be quite plain: GOODSPEED used in repeating the experiments not a single of the 21 American races I mentioned, but 5 others, so that practically we cannot speak of contradictory results.

Yet GOODSPEED's criticism has led me to resume the research on a larger scale. From the 139 samples now examined, among which there was no progeny of the 51 of 1916, 10 germinated badly, so that they must be left out of account. The remaining 129 belonged to the following species, varieties and races: *N. acuminata Hook, affinis, alata L. et O., alata L. et O. var. grandiflora Comes, atropurpurea grandiflora, auriculata Bertol., campanulata W. K., cerinthioides Horn., chinensis Fisch., fragrans Hort., frūticosa L., glauca R. Grah., glutinosa L., Langsdorffii Schrank., longiflora*

Cav., macrophylla Spreng., macrophylla gigantea, paniculata L., petiolaris Schlecht., Plumbaginifolia Willd., rustica L. (7 samples), *Sanderae Hort., silvestris Spreng., solanifera Walp., solanifolia Walp., spomosiflora Dun., suaveolens Lehm., Tabacum L.* (10 samples and further the culture-races: Amersfoorter, Cavalla, Cuba, Deli, Hatano, Kedoe, Pajakombo, Samson, Smyrna, Tombak, Trafasund, Vorstenlanden and Xanthi), *virginica Agard., viscosum Lehm.*

The seeds originated from: Bordeau, Bucharest, Cluj, Delft, Dyon, Genoa, Göttingen, Jassy, Kew, Königsberg, Maltha, Montpellier, Naples, Palermo, Wilna, Wageningen, Xanthi, Zürich, Palestina, Soechoem (Trans Caucasia) and the Dutch Indies (Djember, Klaten, Medan, Pajakombo).

During these experiments I had the great advantage of working in the plant-physiological laboratory of Prof. BLAAUW, who was kind enough to put two rooms of a constant temperature at my disposal. In one of them kept at 20° C. stood a thermostat heated to 26° C., in which the Petri-dishes containing the seeds for the experiment in darkness were put in light-proof boxes, every lot consisting of 200 seeds of each sample. As a rule the temperature-oscillations amounted to some tenths of a degree. The seeds which were to be exposed to the light, were put in the other room on four tables each with 5 rows of 5 dishes (likewise 2 either of them containing 100 seeds of each sample) illuminated by a lamp with a parabolic reflector 115 cms. above the centre of each table. As the lamps raise the temperature, forming part of the heating-system, they were initially kept burning constantly. By turns a 12 hours' exposure to light was succeeded by a 12 hours' darkening by means of a green cloth folded double on top of the dishes (without touching them). As a rule the temperature was 24 to 26° C., it never fell below 21.9 nor exceeded 27.9° C. (the thermometer on moist filter-paper in a Petri-dish on one of the tables). Since oscillations could not be prevented (which did not matter in these experiments as they were only meant as a control on the germination in darkness), the light was afterwards put in and out by means of an electric clock of SAUTER every 12 hours. 100 Watt's Argenta lamps appeared to be too strong for this arrangement, e.g. for the *rustica* from Kew, which germinated more rapidly in darkness. So they were exchanged for 40 Watt Argenta.

Controlling the seeds exposed to light occurred in diffuse day-light, that of the seeds kept in the dark in a dark room by a dark brownish-yellow 16 Watt photo-lamp, as far as I could observe without detriment, though theoretically red or blue light would have been preferable (KINZEL 1907).

No difference after an exposure to light or dark, or a difference of at most 5 % in 14 days was found in: *N. auriculata, chinensis* from Delft, Jassy and Wageningen, *fruticosa, glutinosa* from Naples, *macrophylla gigantea, petiolaris* from Delft, *Plumbaginifolia* from Bucharest and Wageningen, *rustica* from Cluj, Wageningen and Zürich, *silvestris* from Bucharest, *suaveolens* from Delft and Jassy, *Tabacum* from Cluj, Genoa, Naples

and Zürich (and from the *culture-races* germinated equally good without light: Cavalla, Cuba, Samson, Smyrna, Tombak, Trafasund and Xanthi), *texana* from Delft and Naples, *viscosa* from Cluj.

Speaking only of need of light, if the difference between light and dark amounts to at least 20 % (which is of course arbitrary and purposely exaggerated), we must consider as absolute or partial light-germinators: *N. alata* from Delft and Maltha (difference between light and dark 36 resp. 79 % in 14 days), *fragrans* (diff. 33 %), *glauca* from Cluj, Dyon, Maltha, Montpellier and Naples (d. resp. 50, 31, 35, 21 and 36 %), *glutinosa* from Cluj (d. 69 %), *Langsdorffii* from Cluj (d. 77 %), *longiflora* (d. 51 %), *macrophylla* (d. 75 %), *paniculata* from Cluj and Dyon (d. 39 and 34 %), *rustica* from Bucharest and Dyon (d. 36 and 24 %), *silvestris* from Cluj, Wageningen and Zürich (d. 60, 84 and 32 %), *solanifera* (d. 89 %), *Tabacum* from Bucharest and Dyon (d. 26 and 39 %) and the cultivated races Amersfoorter from Wageningen (d. 40 %) and Pajakombo from Western-Sumatra (d. 90 %), *texana* from Wageningen (d. 87 %), *virginica* from Bucharest and Maltha (d. 30 and 75 %).

In both series the species: *glutinosa, rustica, silvestris, Tabacum* and *texana* occur. Perhaps *chinensis, petiolaris* and *suaveolens* might be reckoned among them, but in that case the decisive limit should be reduced from 20 to 10 %.

With respect to *Tabacum* this is a corroboration, made a little less urgent, since with regard to two *Tabacum*-races, Mexico Santiago and Mexico Compostella, BUSSE (1925) stated the need of light, viz. 22 and 73 % with light and 0 and 18 % in darkness. Now it appears, that likewise *rustica*, as to which I initially agreed with GOODSPEED, that it could do without light, be it within narrower bounds, also varies (table I).

Part of the seeds kept in the dark, which remained considerably behind those exposed to light, were subjected to alternating exposure after 14, 21 or 30 days. It usually appeared, that the imbibed seeds had only waited for light and the germination-figure rose suddenly, sometimes even above that of the directly exposed control-seeds. For some lots however the abode in moist air in a constant temperature without germination had proved injurious. The retrogression amounted to 10 to 87 % (e.g. *texana* table I).

Experiments with pure lines.

By far the greater part of the samples discussed concerns seed the genetic purity of which is unknown. An exception are some 8 samples of the 2nd to 5th generations after self-pollination at Wageningen, viz. *Tabacum* Amersfoorter D_5, *rustica* D_4, *atropurpurea grandiflora* D_5, *chinensis* D_3, *macrophylla gigantea* D_1 and D_2, *Plumbaginifolia* D_4, *silvestris* D_4 and *texana* D_3. As to the rest BUSSE's objection to my public-

TABLE I. Germination-percentage of some samples of seed of species and varieties of the genus Nicotiana. [1])

Species and variety Origin	light dark	Germination-percentage after .. days								
		3	5	7	10	14	18	21	25	30
N. alata L. et O., Delft	l	90	.	.	.	90				
	d	1	48	54	.	54 L	67	67	.	67
„ Maltha	l	—	—	3	46	79	.	84	.	84
	d	—	—	—	—	— L	64	71	73	74
N. glauca R. Grah., Cluj	l	—	33	39	45	50	51	53	.	55
	d	—	—	—	—	— L	62	64	65	66
„ Maltha	l	1	87	88	91	92				
	d	—	—	4	32	57	.	60 L	86	88
N. glutinosa L., Cluj	l	43	73	77	79	79				
	d	—	—	3	6	10 L	91	92	.	92
„ Naples	l	60	.	61	62	63		63		
	d	57	63	65	.	65				
N. paniculata L., Cluj	l	85	87	89	.	90				
	d	1	9	36	49	51 L	89	94	.	94
„ Montpellier	l	78	84	.	86	90				
	d	44	80	84	.	84				
N. rustica L., Bucharest	l	59	62	.	64	64		64		
	d	21	26	28	.	28		29		
„ Dyon	l	87	91	92	.	93				
	d	34	58	66	68	69 L	79	80	.	81
„ Zürich	l	91	.	92	.	92				
	d	89	91	.	.	91				
N. silvestris Comes et Speg., Cluj	l	1	26	46	57	60	.	61		
	d	—	—	—	—	— L	13	79	86	87
„ Jassy	l	86	97	99	.	99		99		
	d	3	53	88	91	93				
N. Tabacum L., Dyon	l	19	84	88	90	91				
	d	—	4	31	41	42 L	82	90	91	91
„ Amersfoorter D_5	l	47	88	.	.	89				
Wageningen	d	1	35	48	49	49 L	67	84	.	85
„ Pajakombo	l	—	76	87	90	90				
Western-Sumatra	d	—	—	—	—	— L	79	84		
„ Smyrna	l	82	90	.	91	91				
Palestine	d	3	85	92	94	94				
N. texana Hort., Delft	l	93	.	.	.	93				
	d	23	97	.	.	97				
„ D_3, Wageningen	l	89	90	.	.	90				
	d	—	3	.	.	3 L	.	.	.	3

[1]) L in column 14 days means: transmitted to light.

ation of 1916 holds good, that it is unknown, whether the material is homozygous or not. Now lines from four races of *N. Tabacum* have been examined: Deli, Vorstenlanden, Kedoe and Hatano and the result of the experiments with the pure lines is perfectly parallel to those of the samples perhaps not pure from a genetic point of view. With regard to the term "pure lines" it should be remembered that the purity was judged before from various morphological characteristics, so that the possibility remained, that those races were not pure as to the need of light for the germination, a fact which had not been taken into account. This, however, did not appear to be the case, for in two and three generations the phenomena remained constant, if only the degree of ripening was identical. The following was observed:

A. Deli. *a.* Of a half-chlorina-line the seed of two chlorina-individuals was examined, a D_2 from Sumatra and a D_3 from Wageningen at the same time with an extracted green individual D_2 from Sumatra. In darkness the germination-figures for half-chlorina were 11 and 10 %, of the green plant even 20 %, exactly the maximum found for Deli in 1916.

b. Of the Kloempang-line, the heterozygotes of which always produce ¼ plants of normal height, ½ halfhigh and ¼ dwarfs, the two homozygotes have been examined. The seed of the normal type germinated in darkness in 14 days for 18½ % (in the light 92 % in 7 days) and the seed of the dwarf in darkness for ½ % (in the light 61 %).

c. The deformis-line was very "hard". The *deformis*, *deformis*-heterozygote and the normal type germinated in darkness for ½, 0 and 2 %, in the light in the same order of succession: 98, 91 and 99 %.

In all three lines it is the normal form which germinates least unfavourably without light:

half-chlorina-line: green 20 %, half-chlorina 11 and 10 %,
Kloempang-line: high 18½, dwarf ½ %,
deformis-line: normal 2 %, *deformis*-heterozygote 0 %, *deformis* ½ %.

B. Vorstenlanden ("the Principalities"). Of this I possess 3 lines, two homozygous double-flowered and a "doorschieter" ("shooter") or giant-line, apparently heterozygous for the number of leaves. The two double-flowered specimens germinate excellently in darkness, only initially somewhat slowlier, but within the first week the difference with the seeds exposed to light had disappeared. This is remarkable, as, but for the double flowers and some features connected with them, this tobacco shows much resemblance with the Canari, for which RACIBORSKI stated a need of light. The seeds of the shooter germinated very slowly, also exposed to light. In darkness however nearly half the seeds had shooted within a fortnight.

C. Kedoe. Seed of mutant C of ARISZ germinated without light in a fortnight for 65 %. A subsequent exposure had practically no effect, viz.

TABLE II. Germination of some pure lines of Nicotiana Tabacum L. (Deli, Vorstenlanden, Kedoe, Hatano).

Race, cultivated in .... year ...., generation ...	light dark	Germination-percentage after .. days										
		3	5	7	10	14	18	21	25	30	35	40
A. Deli												
a. half-chlorina-line												
half-chlorina D_2	l	31	80	89	92	92						
Sumatra 1920	d	—	—	3	11	11 L	91	92				
the same D_3	l	19	79	82	.	83						
Wageningen 1922	d	—	2	7	9	10 L	82	84				
extracted green D_2	l	27	82	88	92	92						
Sumatra 1920	d	—	1	10	19	20 L	90	91				
b. Kloempang-line												
dwarf D_2	l	34	51	57	60	61						
Wageningen 1922	d	—	—	—	—	— L	64	65				
high D_2	l	64	91	92	.	92						
Wageningen 1922	d	—	—	13	18	18 L	86	90	91	91		
c. deformis-line												
deformis D_3	l	71	97	98	.	98						
Wageningen 1922	d	—	—	—	—	— L	96	97				
deformis-heterozyg.	l	84	90	.	.	91						
F_4 Wageningen 1923	d	—	—	—	—	— L	87	90	.	90		
extracted normal	l	45	97	99	.	99						
D_2 Sumatra 1916	d	—	—	1	2	L	96	96				
B. Vorstenlanden												
a. *double-flowered line*	l	—	63	96	97	97						
D_x Klaten n^0. 99 1917	d	—	3	94	98	98						
the same D_{x+1}	l	41	59	65	72	73	.	73				
Wageningen 1924	d	6	52	64	68	72	.	73				
b. double-flow. line D_x	l	—	74	90	94	94						
Klaten n^0. 120 1917	d	—	—	87	94	96						
the same D_{x+1}	l	45	66	68	69	71	72	72				
Wageningen 1924	d	26	58	72	74	75	76	76				

TABLE II. Germination of some pure lines of Nicotiana Tabacum L. (Deli, Vorstenlanden, Kedoe, Hatano). (*Continued*).

Race, cultivated in .... year ... generation ...	light dark	Germination-percentage after .. days										
		3	5	7	10	14	18	21	25	30	35	40
the same D_{x+2} Nov.	l	35	80	83	84	85						
Wageningen 1925 „	d	12	27	54	62	63 L	80	84	85	85		
Jan.	d	7	24	35	58	62	.	62 L	86	91	.	91
c. „*Shooter*"	l	—	20	46	55	59	61	62	.	62		
Klaten 1919	d	—	—	23	41	46	48	48 L	53	53		
C. Kedoe												
mutant C of Arisz	l	3	69	79	82	82						
D_x Djember	d	—	—	52	62	65	.	65 L	66	67		
the same D_{x+1} Nov.	l	10	59	78	96	96						
Wageningen 1925 „	d	—	—	1	3	3 L	93	94				
Jan.	l	4	12	17	20	22	23	23	23	76	95	97
„	d	1	1	2	3	4	.	4 L	94	97	.	97
D. Hatano												
D_x	l	73	97	.	.	97						
Wageningen 1923	d	—	1	9	35	56	69	70 L	95	97		
D_{x+1} Nov.	l	15	76	87	90	91						
Wageningen 1925 „	d	—	—	2	3	3 L	89	90				
Jan.	l	4	30	39	50	64	65	65	87	95	.	96
„	d	—	—	—	1	1	.	1 L	92	93	.	93

but 2 %, though on exposure from the beginning 82 % germinated. This Kedoe-line therefore is not so indifferent to light as the two double-flowered lines of the Vorstenlanden ; its behaviour is more parallel with the shooter's.

D. Hatano. This does not germinate in darkness quite so well as the Kedoe-mutant, viz. 56 % in a fortnight, 70 % in 3 weeks, compared with 97 % in 5 days in the light.

Especially the results with the pure lines (table II) have in my opinion proved, that in the "species" *Tabacum* there are very great differences as to the need of light for germination and that those differences are hereditary.

Influence of the degree of ripeness. The results obtained with seed of the Vorstenlanden, Kedoe and Hatano, gathered in September 1925 and

examined in November next following, deviated in so far from those with seed of 1917, 1923 and 1924, that without light the germination was distinctly less favourable in seeds of Vorstenlanden and almost absent in Kedoe and Hatano. As this phenomenon also occurred in the hybrids to be discussed presently, a further research was necessary. From my practise at experiment-stations I knew, that Deli-seed, still a little fresh, does not germinate so well as seed at least six months old. Now it appeared, that seed not yet quite mature is less capable of germinating without light (Vorstenlanden 22 % less), or has not got this capacity at all (Kedoe and Hatano both 3 % compared with 96 and 91 % with light). This proves that a type indifferent to light has lived through a stage of much greater need of light, in other words that the forms that always need light for germination, do not reach the indifferent stage in after-ripening. And this renders the fact a little more comprehensible that we find all kinds of transitions between the absolute need of light and the absolute doing without. (How the seeds behave directly after they have been gathered has still to be investigated into).

On repeating the experiments in January I found, that on the whole the seeds were even more latent, that also in the light the progress of the germination was much slower. In March they recovered, i.e. with regard to the germination with light, not yet for the germination in darkness.

The crosses.

Crosses between *N. Tabacum* Amersfoorter (half-"light-germinator") and an indifferent *N. rustica* and of Amersfoorter with *silvestris* ("light-germinator") did lead to a flowering F_1, but in both cases a sterile one, likewise on backcrossing with the parents.

Better results were obtained by crossing some *Tabacum*-races the one with the other. The double-flowered lines of Vorstenlanden (indifferent) were reciprocally crossed with the *deformis*-heterozygote (pronounced "light-germinator"). With Klaten N°. 99 germination-figures of 0 and 4 % were obtained, with Klaten N°. 120 on the contrary 84 and 87 %, though we observed a vastly slower progress than with the two parents. In darkness however it was but 1½ and 0 % after 30 days, i.e. the need of light is dominant and indifferent is recessive. The results of the two reciprocal crosses of Vorstenlanden and Deli being identical, the seed-coat has no influence and the need of light is a characteristic of the embryo.

The F_2 behaved differently, viz. 36, 22 and 9 % germinate in darkness in a fortnight to 96, 86 and 63 (after 45 days 87) % in the light. The differences of 27 and 13 % can be the result of an unequal degree of after-ripening and do not prove anything with respect to the ratios, nothing therefore as to the number of factors to which the difference is due. A repetition of the germination-experiment in January yielded 9, 14 and

1 % germination in a fortnight and after 21 days exactly the same figures. Yet the germinative-power had not got lost, for when after 21 days the seeds were exposed to light, a total amount of 95, 90 and 72 % had germinated after 19 days.

TABLE III. Germination of seeds of the F_1 and F_2 after crossing a "light-germinator" with an indifferent race and a "light-germinator" with a half-indifferent race

Gen.	Parents. Year of harvest	light dark	Percentage of germination .. days														
			3	5	7	10	14	18	21	25	30	35	40	45	50		
F_1	*deformis*-heterozygote × Vorstenlanden Klaten No. 120, 1924	l	—	6	18	31	32	.	32	43	44	64	84	.	84		
		d	—	—	—	1	1	.	1	1	1 L	83	86	.	86		
F_2	F_1 plant with sessile leaves 1925 Nov.	l	18	94	95	96	96										
	"	d	—	3	29	36	36 L	93	97	.	97						
	January	d	—	1	5	9	9	.	9 L	87	94	.	95				
F_2	F_1 plant with petiolate leaves 1925 Nov.	l	14	81	84	85	86										
	"	d	1	5	16	22	22 L	83	90	92	94						
	January	d	—	2	10	13	14	.	14 L	72	85	90	90				
F_1	Vorstenlanden Klaten No. 120 × *deformis*-heterozygote 1924	l	—	3	.	4	5	9	32	47	60	85	85	.	87		
		d	—	—	—	—	—	—	—	—	— L	83	86	.	86		
F_2	F_1 plant with petiolate leaves 1925 Nov.	l	8	49	53	61	63	67	77	78	80	85	86	87	87		
	"	d	1	2	7	9	9 L	60	75	78	79	82	82	87	87		
	January	d	—	—	1	.	1	.	1 L	48	55	69	72	.			
F_1	*deformis*-heterozygote × Hatano 1923	l	79	94	97	.	97										
		d	—	—	9	14	14 L	95	98	.	98						
F_2	F_1 plant with petiolate leaves 1924	l	71	92	.	94	95										
		d	—	—	4	8	8 L	86	94	95	95						
F_2	F_1 plant with petiolate leaves 1924	l	64	94	96	.	97										
		d	—	—	2	5	5 L	95	97	.	97						
F_2	F_1 plant with winged leaves 1924	l	91	98	.	.	98										
		d	—	5	55	68	70 L	97	97	.	97						

Likewise with the F_1 of *deformis*-heterozygote and Hatano (half-

indifferent) it appears, that the need of light is dominant: in darkness 14½ % in a fortnight, in the light 97 % in 7 days. The dominance therefore is not so absolute here as with the crossing with the Vorstenlanden, though the Hatano-line approximates the Deli nearer than the Vorstenlanden-line as to need of light (table III).

The F_2 of *deformis*-heterozygote × Hatano segregated likewise and as the seed is of the year 1924, the figures ought to be trustworthier. Seed of three, F_1 individuals germinated in 14 days in darkness for 8, 5 and 70 % (in the light 95, 97 and 98 %). The first two F_1 individuals had petiolate leaves (characteristic of heterozygosity for the *deformis*-factor and *deformis* is very "hard" without light), the third had winged petioles (Hatano-characteristic and Hatano germinates fairly well in darkness, 56 % in 14 and 70 % in 21 days). This association of characteristics may be accidental, but it may also point to coupling (though the chance of finding this with a species with haploid 24 chromosomes is not very great). A large number of F_1 individuals will only be decisive. It is true seed of seven F_2 plants has been examined, but that number too is much too slight; moreover this seed of 1925 does not yet give any constant results. For the present it is only a fact, that segregation occurs in the second generation; whether this is Mendelian segregation is an open question.

May be JOST's saying (Pflanzenphysiologie II, p. 131, 1923): "Somit bleibt von dem Begriff "Lichtkeimer" nicht mehr viel übrig", is perfectly correct, though it has still to be proved, if in every case light may be replaced by chemical reagents. Experiments to make my "hardest" seed, the F_4 *deformis*-heterozygote germinate in darkness with diluted acetic acid, nitric acid, potassium nitrate and lime have proved a failure. With the answer to the question whether the idea "light-germinator" is correct or not, the difference existing between the so-called light-germinators and the indifferent tobacco-types does not stand or fall. That difference is a fact, is hereditary.

BUSSE tried to give an explanation for the differences in need of light for the germination stated by him viz. as a result of natural or artificial selection through climate (intensive sun-light in the tropics) or cultural methods (covering the seed-beds with a coat of soil at least 1 cm. thick in South-Eastern Europe and Asia Minor). That sounds natural and plausible. Yet this explanation is presumably incorrect, for all four of my Java types germinated without light either excellently or for nearly 50 % (98, 96, 46 and 65 %), a result totally deviating from what RACIBORSKI and JENSEN found for tobacco of the Principalities, while there is not a single reason for the least doubt about the exactness of their communications. That we should not generalise with regard to this material is obvious.

LITERATURE.

To the literature cited by BUSSE (Ztschr. f. Bot. Bd. 18, 1925) should be added:

ARISZ, H. W. Over mutaties bij tabak. Tweede Nederlandsch-Indisch natuurwetenschappelijk congres, Bandoeng, 1922.

GOODSPEED, T. H. Notes on the germination of tobacco seed III, Univ. of California Publ. in Botany V, p. 451—455, 1919.

HONING, J. A. Nicotiana deformis n. sp. und die Enzymtheorie der Erblichkeit. Genetica V, p. 455—476, 1923.

Botany. — *"On new phenomena caused by irritation of roots."* By Prof. J. M. JANSE. (Communicated by Prof. J. C. SCHOUTE.)

(Communicated at the meeting of March 27, 1926)

During a recent research of movements effected by excitation of roots, my attention was attracted by some phenomena unknown to me, which therefore I studied more in detail.

The first experiments were made with seedlings of *Sinapis alba, Helianthus annuus, Pisum sativum* and *Lupinus luteus,* the plants mostly used at the beginning of experiments. Afterwards I examined the radicles of some other plants, in order to test the universality of the phenomena observed. I did not select special plants, but made me give different seeds from the Botanical Garden. Yet all gave fully corresponding results, though there were differences in sensibility and not all were equally adapted for the purpose, be it for accessory reasons. *Sinapis, Brassica* and *Onopordon* f.i. turned out to be the most suitable, especially for microscopical examination.

The phenomena observed were the following:

1. If a seedling is pulled up somewhat carelessly out of the soil, the result is sometimes already a certain retardation in growth; the retardation is more important when the friction is augmented in some way. At an experiment with a radicle of *Helianthus annuus,* growing between two slabs of glass and observed with the horizontal microscope, the speed of growth at the beginning, at about 16° C., amounted to 0.33—0.36 m.m. an hour. After having been very strongly excited, by moving it some 200 times between wet fingers, it showed when it had been brought back between the slabs of glass again, a speed of growth 0 during the first 30 minutes, but in the hours following the rapidity increased successively to 0.08, 0.28, 0.23 and 0.30 m.m. an hour, so that after 5 hours the primitive speed was nearly reached again. So the result is, that a strong excitation provokes a very considerable delay in growth, but without causing a permanent disturbance.

2. After effectual friction, e.g. as in the experiment described above, the root turned out to have become a little thinner and weaker over a certain distance (e.g. from 5—9 m.m. from the top). The diameter of a root of *Helianthus* for instance, decreased from 0.66 to 0.47 m.m.

As the length of the growingzone of such a root amounts to about 10 m.m., the emaciation principally takes place in the zones approaching the full-grown state.

3. Another phenomenon, which always accompanies the former, is the fact that the root becomes more or less transparent. As the opaquencess of the normal root is due to the great quantity of air contained in the numerous intercellular spaces, which are principally arranged in longitudial direction,

the transparency of the root is founded on the disappearance of this air, or, as we shall call them for convenience sake, the air-filaments, because they present themselves to a certain extent as such.

Besides it was observed that the disappearance of the air could be studied also at a magnification of 100, and in thin roots even of 250 times. On a transverse section, the intercellular spaces are mostly wedge-shaped, so that, when a rootlet is examined lengthwise, they present themselves some times as very fine filaments, sometimes as broad ribbons, according to their being examined from the narrow or from the broad side; they strike the eye especially by the total reflection of light in the cellwalls surrounding them. The unaltered roots are so opaque, that by means of the microscope only the higher situated filaments are perceptible with some distinctness, but during the process the examination becomes more and more easy, as the tissue grows more transparent.

To observe the phenomena, the best way is to excite not too strongly a thin radicle of *Sinapis, Brassica* or *Onopordon,* of about 1 to 2 c.m. in lenght (it is sufficient to move it from 1 to 5 times between the fingers) and to examine it immediately afterwards in water under a cover-glass.

The first thing to be seen then is that often an important number of air-filaments are no longer limited by straight lines, but that they show at one or both sides small depressions; these depressions deepen by degrees and so reach the other side of the filament, which is in this way interrupted in many places. The number of these intermissions increases hand-over-hand in some minutes; at the same time many of them spread in a lateral direction; by this longer or shorter parts of the air-filaments are occasionally pushed aside (sometimes even with a rather considerable speed) or, and far more frequently, two of the intermissions approach one another, so that the air-bubble lying between them decreases gradually in length and disappears wholly at the end.

From this it follows that the infiltration is caused chiefly by the disappearance of the air from intercellular spaces, and not by a movement or displacement of it to the other, unexcited parts of the plant. The phenomenon seems to appear in every cell rather quickly, so it may be that the alterations observed microscopically originate from the cells reacting somewhat slowly. The place of the air is occupied by a liquid emerging in small drops, which make out the mentioned interruptions, from the adjacent parenchymatous cells. This latter fact could not be ascertained directly, when, as usually, the cells contained an untinged cellsap, but another origin of the drops is unconceivable under these circumstances. Yet I have tried to make sure the origin of the liquid in another way; for those new experiments I chose seedlings of *Beta vulgaris,* namely of a very dark coloured variety of the red beet, as I expected that here the cells of the young parenchyma of the root would contain red cellsap. This supposition, however, turned out to be incorrect, the root being untinged over about 20 m.m. from the top; the anthocyan appears only later on, as soon as the cells

are full-grown and so I had to use full-grown cells for these experiments.

After irritation by effectual friction of these full-grown parts the interruption of the filaments and the dissolution of the air could be clearly observed, even when a magnification of 250 times was used. Even scrupulous examination showed not the slightest difference in tinge between the parts of the intercellular spaces where the air had been expelled and the immediately adjacent cells. So we are entitled to conclude, for the present at least, that the liquid leaving the cells had the same colour as the cellsap itself, so that it consisted indeed of a part of the unaltered cellsap, as was presumed.

The explanation of these phenomena follows here later on.

When the root is excitated feebly, the quantity of air usually disappearing does not amount to such an extent as to render the root wholly transparent ; in this case a strong irritation is necessary to obtain this phenomenon.

This sensibility of a distinct zone of young roots is, as is evident from the accessory phenomena, quite different from the one observed by DARWIN, especially because of the fact that the latter was proved to reside only in the top of the root.

4. When the plants, after having been irritated as described above, are cultivated, e.g. between two slabs of glass, they show after the lapse of a day still another phenomenon, consisting in the fact that the root, which has grown in length more or less, shows a local thickening. This swelling always appears in a corresponding place, i.e. the place occupied the day before by the not yet grown-out cells of the zone of growth. This swollen part occurs always under the part emaciated by the excitation, which at this time is mostly but little restored, and contains, accordingly, only a small quantity of air. The cells that were at the beginning of their development had therefore not increased as much in length, as otherwise would have been the case, but on the contrary their radial dimension had become augmented importantly. This phenomenon can be observed easily after they are brought in alcohol, and then also the fact can be observed that celldivisions have played no part at all in this thickening. Below the thickened part the root slowly resumes its cylindrical form and normal dimensions.

The importance of this modification of the dimensions of the cells in the swollen part is shown in the subjoined table, derived from an experiment where *Sinapis* was used ; the dimensions of the cells in the narrowest and the thickest part of the root amounted to :

	narrowest part.	thickest part.	proportion
longitud. dimens. :	0.35 m.m.	0.14 m.m.	1 : 0.4
radial dimens. :	0.19 m.m.	0.28 m.m.	1 : 1.5

When we presume, as we are entitled to, that the tangential dimension of the cells has remained unaltered, we find that the areas and consequently the volumes of these cells bear a proportion of 10 : 6, so that the local

swelling is accompanied by an important staying behind of the volume of the cells.

The degree of swelling is very different according to the plants examined; it depends upon the degree of excitation and may reach a considerable amount as is evident from the subjoined table indicating the diameter of the thicker and of the narrower part (the latter being the part excitated the day before) of the roots, one day after the excitation :

	narrower part	thicker part	difference
Sinapis alba	0.43 mm.	0.85 mm.	0.42 mm. or 98 %
Helianthus annuus	0.85 „	0.95 „	0.10 „ „ 12 „
Pisum sativum	0.69 „	0.95 „	0.26 „ „ 38 „
Lupinus luteus	0.90 „	1.30 „	0.40 „ „ 44 „
Onopordon bracteatum	0.44 „	1.58 „	0.14 „ „ 32 „
Mirabilis Jalapa	0.57 „	1.14 „	0.57 „ „ 100 „
Impatiens glanduligera	0.57 „	0.71 „	0.14 „ „ 24 „
Euphorbia Lagascae	0.65 „	0.90 „	0.25 „ „ 38 „
Nigella aristata	0.38 „	0.52 „	0.14 „ „ 31 „
Beta vulgaris	0.40 „	0.59 „	0.19 „ „ 47 „

The reaction comes to an end within 24 hours, as is proved clearly by the fact that after that lapse of time, a renewed friction provokes a second swelling, etc. In this way a root of *Lupinus*, excited on three consecutive days, showed therefore 3 successive swellings at a distance of some millimeters one from the other ; the growth amounted during the day after the second excitation to 6.5 m.m. after the third to 4.25 m.m., but the following day, when the rootlet had not been excited previously, the growth amounted to 9¾ m.m., showing thus again that the excitation retards the rate of growth.

When, after having been excited, the plants are cultivated in humid air, the fact can be observed that in the narrow part, which is situated immediately below the old roothairs, no new hairs are produced, but that they reappear quite normally at the beginning of the swollen part. From this might be concluded once more that especially the nearly full-grown cells undergo the influence of the irritation, for the fact that the old root-hairs are all situated above it, proves that the cells situated in this zone were full-grown already when the excitation took place ; indeed roothairs are never formed in the zone of growth.

Now the question arises how to explain the phenomena described.

1. The direct examination showed that the interruption of the

airfilaments was due to small drops of liquid, penetrating into the intercellular spaces. It is evident that those drops can only come from the adjacent parenchyma-cells, so that they must consist of pure water or of cellsap. Theoretically only the last hypothesis is acceptible for reasons, which I developed circumstantionally in a former article [1]) concerning the irritations of *Mimosa*, which are probably identical with those here described. The principal reason was, that in the case of pure water being expressed, the cellsap had necessarily to be divided very quickly into water and a more concentrated solution, which is absolutely inacceptable, among other reasons because of the very strong attractive power, which these liquids excercise on each other.

When indeed a part of the cellsap is expressed this only could have taken place after the wall of the vacuole, having lost its semipermeability, has become permeable for all substances contained in the vacuole; only after this has happened the elastically stretched cellwall will be able to press part of the cellsap into the intercellular spaces. The emission of coloured cellsap as described in the case of *Beta* would have been the best proof that indeed the wall of the vacuole had become entirely permeable, because of the fact that it then must have allowed even the large molecules of anthocyan to pass.

Besides this the last named effect proves that the cells which are just full-grown, may react in the same way as the nearly grown-out elements of the growing-roots, which affords a motive to extent this researches also to full-grown tissues and to those of other plants.

2. A part of the cellsap thus arriving under a certain pressure in the intercellular spaces, will exercise a pressure on the air contained in them. When an air-filament could easily be moved from its place (as is sometimes observed) this would undoubtedly take place everywhere, so that at the end all air would be driven into the unexcited parts of the plant. But, as the resistance in the narrow intercellular spaces is too great for this, this pressure will compress the air contained in them till an equilibrium has been established between this pressure and the rest of the tension of the cellwalls, these cellwalls being then still — be it to a lower degree than before — elastically stretched. The fact that they are still stretched to some extent is evident, as the excited root has kept some turgescence.

The partial or total disappearance of the air from the intercellular spaces, as is observed by means of the microscope, is therefore due to the air being dissolved, as the solubility of gases in water is proportional to their pressure; so this phenomenon, however striking it may be, is not of any theoretical importance.

3. The decrease of the ration of growth of the roots after the irritation is evidently due to the diminishing of the turgor of the still growing cells, as

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[1]) J. M. JANSE, Ueber die Permeabilität des Protoplasma. Verslagen en Mededeelingen der Kon. Ak. van Wetenschappen, Amsterdam, Afd. Natuurkunde, 1888, Dl. IV, p. 410.

the ration of growth is principally determined by this tension. The fact that the ration of growth increases afterwards again slowly to the primitive value, cannot be explained in so simple a way, as certainly different circumstances, not all entirely known, play a part therein. But we may, however, remark here, that the nearly full-grown cells which had lost a part of their cellsap do not take it in again so easily, as the observation showed that the day after their irritation the radicle had not reassumed its former dimension, nor the air had returned completely into the intercellular spaces.

4. The cause of the modification in growth of the cells of the younger zones of the root consisting in a diminishing in the longitudinal and an increase in the radial direction can not yet be explained. Nevertheless we may remark here that in various other cases already described, a decrease of growing in length taking place for some reason or other was always accompanied by an important development in thickness. Such a case was described already by VÖCHTING in the report of his experiences with *Papaver* [1]). When a nearly full-grown flowerbud(which was still hanging down) of *Papaver Rhoeas* or of other species of this genus was prevented from erection by means of a weight attached to it, it showed soon a diminishing of growth and at the same time a abnormal swelling of the upper part of the stem ; VÖCHTING adds that every hindrance of growing in length can cause an abnormal increase in diameter without new celldivisions taking place.

A corresponding phenomenon can often be observed in geotropic curvatures, especially of seedlings ; it can appear in the root as well in the hypocotyl ; new experiments, with roots of *Pisum* and *Lupinus*, and roots and hypocotyls of *Sinapis* convinced me again. Though these swelling must have been seen often, and by many investigators, so that they can even be recognised in some illustrations, they have apparently always escaped special attention.

When we assume that even the normal vertical growing in length might be considered as a reaction upon the irritation due to gravitation (as to me it seems probable [2])) it follows from this, that in the both cases mentioned this irritation by gravity must have decreased, so that we might conclude that the diminishing of this excitation, causing a decrease of the growing in length, could be held responsible also for the stronger growing in thickness. This fact would be of still more importance, if as I can yet but suppose, inversely a swelling, appearing under circumstances of the same kind would indicate a previous decrease of the excitation by gravity, as this would provide us with a direct and permanent indication of a diminishing of the intensity of this excitation having taken place.

BLAAUW has shown by numerous experiences, performed with extremely sensitive instruments, that strong light can slacken also the growth in length;

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[1]) Die Bewegungen der Blüthen und Früchte, Bonn, 1882, p. 120—124.

[2]) J. M. JANSE. On stimulation in auxotonic movements. These Proceedings 26, p. 171.

it was therefore that I wished to see if the phenomena described above could be observed also in this case; this indeed proved to be the fact. One of the experiments performed in order to solve this question was the following:

Fourteen seedlings of *Sinapis*, of 20—40 m.m. in length, were exposed simultaneously during 65 minutes to the light of an Osramlamp of 1200 candles, placed at a distance of 24 c.m.; a little mirror was placed at the other side of the rootlets. After the exposure all the roots, which contained before the ordinary quantity of air in their intercellular spaces, had become transparent, which was, here too, due to a considerable diminishing of the air in them, as was proved by the microscopical examination.

After this the plants were cultivated in vertical position between two slabs of glass, separated by strips of thick blotting-paper. After 6 hours no growth worth mentioning could be observed, but after 24 hours the length had increased with 2—11 m.m. So the growth had been strongly disturbed, it is true, but the roots had not suffered any permanent injury by the exposure; roothairs too had been developed here, and just in the same way as in the case of irritation by friction; moreover the next day the swellings appeared very plain occurring also at the corresponding places. The result of the measurements was the following (the roots have been tabulated after their decreasing growth in length):

No.	1	2	3	4	5	6	7	Average
Diam. of the narrower part in mm.	0.54	0.54	0.54	0.50	0.57	0.63	0.60	0.56 mm.
Diameter of the thicker part in mm.	0.65	0.69	0.63	0.63	0.72	0.79	0.76	0.70 „
Increase of diam. in mm.	0.11	0.15	0.09	0.13	0.15	0.16	0.16	0.14 „
Increase of diameter in %	20	28	17	26	26	25	27	24 %
Growth in length in mm.	11.0	11.0	10.0	10.0	10.0	10.0	7.5	10.0 mm.

No.	8	9	10	11	12	13	14	Average:
Diam. of the narrower part in mm.	0.60	0.54	0.63	0.63	0.66	0.57	0.60	0.61 mm.
Diam. of the thicker part in mm.	0.76	0.76	0.95	0.77	0.76	0.80	1.01	0.83 „
Increase of diam. in mm.	0.16	0.22	0.32	0.14	0.10	0.23	0.41	0.22 „
Increase of diam. in %	27	41	51	22	15	40	68	38 %
Growth in length in mm.	6.5	6.5	5.0	4.5	4.0	3.0	2.0	4.5 mm.

From this table it is evident that everywhere a plain perceptible swelling was observed, which was by no means less important than in the case of friction. When we compare the 7 plants which had grown most (n°. 1—7) with those growing more slowly (N°. 8—14) we see that, on an average, the former show a growth of 10 m.m. and an increase of diameter amounting to 24 percent, while the later growed in average 4.5 m.m. and increased in diameter 38 percent. So swelling was the more important as the slackening in growth was more considerable. Now and again trifling curvatures were observed, owing to an inevitable difference in excitation at the different sides; the curvature was always situated in the swollen part.

From these results it is evident that the reaction of the roots upon friction and upon exposure to intense light were so fully corresponding in all respects, that we may deduce from it that the delay in growth observed by BLAAUW, and which he calls „lightgrowingreaction", is due just as well to the becoming permeable of the wall of the vacuole of the cells; a microscopical examination of his radicles would certainly have shown him also the more or less complete infiltration of their parenchyma.

This result must give a fully different insight into the importance BLAAUW attributed to his „lightgrowingreaction" as an explanation of the cause of heliotropic movements, as this reaction might be regarded now as a more or less important disturbance of the process of growth, more than as a regulation of this process in behalf of a heliotropic curvature.

From the facts enumerated it has become probable that other stimulations too might provoke the same effects, but this can only be established by further researches.

It seems to me that the temporary permeability of the wall of the vacuole under the influence of a more or less effectual touch is also found in many other plants, for different phenomena have been observed and also described, which are certainly due to such a modification of the properties of the wall of the vacuole.

In first instance this may be the fact with the movements of the leaves of *Mimosa pudica*, as was already remarked before, but besides that there are other leaves able to execute similar though less obvious movements, and furtheron there are many young shoots, etc. that modify temporarily their form after friction or that are bending, also because they loose part of their turgor. I can but mention here the many experiments made by HOFMEISTER [1]) and by PRILLIEUX [2]), but must refer for further expositions to another article, which shall be published elsewhere before long.

These latter phenomena occur exclusively at just full-grown parts, which show rather soon after the friction etc. a certain curvature accompanied by some diminition in turgor; only after a rather long time (some hours) the

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[1]) W. HOFMEISTER, Ueber die Bewegungen saftreicher Pflanzentheile nach Erschütterung, Jahrbücher für wissenschaftliche Botanik, 1860, Bd. 2, p. 237.

[2]) Ed. PRILLIEUX, Étude sur les courbures que produisent les secousses sur les jeunes pousses des végétaux. Annales des Sciences Naturelles, 1868, 5me Série, T. 9, p. 248.

organs reassume their primitive state. In these cases we might admit probably also that the cellsap which had been expelled before, was absorbed again by the same cells, thus unlike the phenomenon observed here in the case of still growing rootlets, where only still growing cells were concerned. Here too, however, the decision must be left to a further research.

POSTSCRIPT.

As BLAAUW was not able to provoke the „lightgrowingreaction" in *Avena sativa* or *Raphanus sativus*, even with very strong light, I examined these species too with regard to their sensivity to friction. Both proved to be not less sensitive than the other plants used; besides *Raphanus* showed to be very suitable for the microscopic examination of the phenomena described, as e.g. the rubbing between wet fingers, repeated about 20 times, and immediately followed by the microscopical examination gives an excellent example of the direct consequences of excitation.

On the next day all these roots showed the formation of roothairs exactly in the same way as in the other cases described and also the increase in thickness did appear; in the case of *Raphanus* it reached the value of 44 and in one of the cases of *Avena* even that of 153 per cent.

So the roots of *all* kinds examined were sensitive to friction and reacted upon it all exactly in the same way.

Geology. — *"Concerning the Occurrence of Quaternary Corals in Angola."* By F. J. FABER. (Communicated by Prof. G. A. F. MOLENGRAAFF.)

(Communicated at the meeting of April 24, 1926).

In an article by Prof. H. GERTH entitled: "De verspreiding en de jonge ouderdom der Koraalriffen in den Atlantischen Oceaan buiten West-Indië" [1]) the following passage occurs: „Finally I wish still to observe that, to my knowledge, raised coral-reefs do not occur anywhere along the Atlantic coasts of South America and Africa, which must have been the case if quaternary reefcorals had occurred on these coasts as early as the Tertiary or the Quaternary. True, in some places tertiary deposits prevail, but they do not contain reefcorals. This had already been marked by GERTH in a map: "Vorkommen der Riffkoralen im Känozoikum" [2])

The occurrence of cainozoic reefcorals on the Westcoast of Africa is not known beyond a few degrees south latitude, which is pointed out emphatically in the above-quoted publications.

That no corals occur there now, is also owing to the cold ocean-current from the south, which flows along the coast, the so-called Benguela-current, of which the influence can still be observed far towards the north [3]). For their viability the ordinary reef-corals require a temperature of the water of more than 20° C. That corals should not have existed on the Westcoast of Africa throughout the Cainozoicum is *not* the fact.

The sedimentary deposits of the Westcoast of Angola near Mossámedes, which, as a rule, have a horizontal position, are referred to the Tertiary by several researchers [4]); presumably they may even be classed under the young Tertiary. I have no reason to suppose that the rock of Balo do Chapéu Armado, nearly 100 k.m. north of Mossámedes, is of an earlier geological period. Probably a young tertiary coastal fringe extends from

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[1]) Vakblad voor Biologen, 7e jaargang, No. 5, pp. 68—71 (1926).

[2]) Gedenkboek-VERBEEK, edited by the Geol. Mijnbouwkundig Genootschap: H. GERTH, „Die Bedeutung der tertiären Riffkorallenfauna des malayischen Archipels für die Entwicklung der lebenden Riff-Fauna im indopazifischen und atlantischen Gebiet", p. 173—196, Map on p. 191 (1925).

[3]) According to G. SCHOTT the seawater has a surface-temperature only just to the south of the Congo-mouth, permanently higher than 20° C. (Geogr. des Atl. Ozeans 1926 Tafel IX). Already at a little depth does the temperature of the water fall considerably.

[4]) See, amongst other works, J. B. BEBIANO. Geologia e Riqueza Mineira de Angola. Lisboa 1923. p. 60 sqq.

a point south of Mossámedes to a point north of Chapéu Armado. In this rock, which I had an opportunity to examine, no corals were identified.

The coast supplies an impressive example of decomposition and over nearly its whole length it consists of a normal barrier, high 40 m. and upwards, which is continually excavated by the ocean to the effect that, time and again, large blocks of rock break off and slide down. Even though in this region raised, comparatively recent coralreefs did occur, there would be little probability of their not having fallen a victim to the erosion. Only when extraordinary circumstances might have let a space intact, this space could be detected.

The Balo do Chapéu Armado is situated at about 14° 23' south-latitude. It does not cut far into the land and is bounded on the north and on the south by the normal, steep sedimentary barrier, on the east by a beach, consisting of sand carried by a rivulet that has silted up half of the bay. Consequently the steep barrier can be traced a few hundreds of meters landwards. Here then the great marine disintegration had been put a stop to for some time.

J. C. SCHAGEN VAN SOELEN as well as the present writer found fragments of coral with a fresh appearance on a small terrace about 8 m. above the sea-level and another 10 m. higher up in the crumbled detritus covering the lower part of the steep barrier. Presumably these fragments were derived from a white speck on the otherwise yellow to reddish-brown rock on considerably greater height. Whether this is so, or is not, we could not make out, no more could we decide whether the coral had originated from the rock or had grown upon it and would consequently be much younger. The latter circumstance would tally with the fresh appearance and is in my opinion the most plausible view. The district being all but uninhabited and uninhabitable we need not think of any other source and a transport by men.

Prof. GERTH of Leyden was so kind as to examine one of the fragments. He referred the coral to the species *Siderastrea siderea*, which still occurs in the Atlantic ocean and is known already from the younger Tertiary of West-Indies. The other fragments have been formed by the same species of corals.

Very likely the land was upheaved at least 200 m. after the formation of the young tertiary deposits; anyhow up to about this height tertiary rocks have been identified. Probably this movement was rather rapid. In complete harmony with this is the markedly revived character displayed by all rivers not far from the shore. The sea attacked the new shore, an erosion-cliff was soon formed and a condition obtained, very similar to the present. The shore lay generally more towards the west, but the bay of Chapéu Armado had already taken origin. Subsequently a subsidence of the land took place (or an upheaval of the level of the sea), while circumstances prevailed that promoted the genesis of corals. It would seem to me that these animals built up their reefs here in the Pleistocene.

It is still a moot point what we have to think of the Benguela-current in this stage. It is certain that the land rose again above the sea, and that the corals came to lie about 30 or at least 20 m. above the sea-level. The immense marine erosion caused the raised reefs to disappear everywhere. Only in a single protected space, where the sea could not undermine the rocks as it did here near Chapéu Armado, a meagre remnant remained.

Chemistry. — *"Equilibria in systems, in which phases, separated by a semipermeable membrane"* XVI. By F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of March 27, 1926).

Some general considerations.

We imagine in the diagrams of the previous communications to be drawn a straight line through the point W. If this line is situated totally out of the region of dimixtion, then the $O.W.A.$ of the liquids of this line increases, according to those liquids being situated further from the point W. If this line intersects a region of dimixtion of two or three liquids, then the same is true for the parts of this line situated outside this region. If we take f.i. the line Wg (fig. 2 XIII); its point of intersection with curve m_1a_1 is called s. The $O.W.A.$ of the liquids of this line then increases from W to s and from a_2 to g; the $O.W.A.$ of liquid a_2 is larger also than that of liquid s. On the line Wm_1m_2 (fig. 2 XIII) the $O.W.A.$ increases from W to m_1 and further from m_2 towards points, situated further from point W; the $O.W.A.$ of the liquids m_1 and m_2 is the same, however.

As the liquids of a straight line going through point W can arise from one another by taking in — or losing water, we may say therefore:

of two stable substances, which can be converted into one another by taking in — or losing water, the liquid with the smallest content of water has the greatest $O.W.A.$ and the liquid with the greatest content of water has the smallest $O.W.A.$ If, however, the two liquids can be in equilibrium with one another, then they have the same $O.W.A.$ (f.i. m_1 and m_2 in fig. 2 XIII).

This rule is a special case of a more general rule which we shall discuss. We take an equilibrium, in which, at taking in and losing a little water, a phases-reaction occurs, viz. a reaction, at which the quantities, but not the compositions of the phases change. Let us take f.i. the system

$$L_{q1} + L_{q2} + L_{q3} \qquad \text{fig. 1 XV.}$$

Herein when taking in water occurs the reaction

$$L_{q3} + water \rightarrow L_{q1} + L_{q2}$$

and when losing water the reaction:

$$L_{q1} + L_{q2} - water \rightarrow L_{q3};$$

the composition of the phases does not change, therefore, but the quantities change. The same is valid also for the equilibrium of the three liquids q_1 q_2 and q_3 in fig. 2 XV; for the ternary equilibria:

$X + Y + L_c$, $H + Y + L_d$ fig. 2 V; $Y + D + L_c$ fig. 2, 3 V

$X + D + L_d$ fig. 2, 3 V; $F + L_{q1} + L_{q2}$ fig. 3 XV

and for the binary equilibria:

$$H+L_a,\quad H+Y,\quad X+L_b \qquad \text{fig. 2 V.}$$

In all those equilibria viz. at taking in- and losing a little water a phases-reaction occurs; consequently the *O. W. A.* does not change; we shall call similar systems: "osmotic invariant".

In a system of n components, which consists of n or more phases, at taking in- or losing water always a phases-reaction can occur, unless in special cases. If we take f.i. three phases of a ternary system, which can be represented accidentally by three points, situated on a straight line, which does not go through the point W, then between those phases a phases-reaction is possible, but this is not the case when taking in or losing water.

If a system of n components consists of less than n phases, then in this system generally no phases-reaction is possible when taking in or losing water, unless the compositions of the phases satisfy certain conditions. This is f.i. the case in the ternary equilibria:

$$D+L_w,\quad D+L_v \quad \text{fig. 2 IV,} \qquad L_{m_1}+L_{m_2} \quad \text{fig. 2, 3 XIII.}$$

They are, although they consist of two phases only, yet osmotic invariant.

If in a system by taking in and losing water one or more of the phases change their composition, then the *O.W.A.* changes also; we shall call such a system "osmotic variant".

This is the case f.i. with the systems:

$$Y+L_a,\quad Y+L_b,\quad Y+L_c \qquad \text{fig. 1, 2, 3 III}$$

$$H+L_c,\quad H+L_d \quad \text{fig. 1 IV;} \qquad H+L_a \quad \text{fig. 2 IV}$$

$$L_{a_1}+L_{a_3},\quad L_{b_1}+L_{b_2},\quad L_{c_2}+L_{c_3} \qquad \text{fig. 1, 2 XV.}$$

We now take two similar systems of such a composition that the one can arise from the other by taking in or losing water. This is f.i. the case with the systems $Y+L_b$ and $Y+L_c$ in fig. 1 III, if we choose the quantities of the phases in such a way that the points, which represent those two systems, are situated on a straight line going through point W.

If both systems, which we shall call E_1 and E_2, can be accidentally in equilibrium with one another, then they have the same *O.W.A.* If this is not the case, then, if we bring both together, a new system E_3 will be formed. Consequently we have the conversion:

$$E_1+E_2 \rightarrow E_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It now depends on the ratio of the quantities of E_1 and E_2 which system E_3 can be formed. If we take a definite ratio and if we consider stable states only, then E_3 is completely defined. If we now assume that E_1 has a greater amount of water than E_2, then appears from (1) that E_3

has also a greater amount of water than E_2 but smaller than E_1. Consequently we can have the conversions:

$$E_1 - water \rightarrow E_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$E_2 + water \rightarrow E_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We now form the osmotic system:

$$E_1 \mid E_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

As herein by diffusion of water at both sides of the membrane the system E_3 can be formed, and as this system, as we assumed above, also is the most stable state, this will really take place. Therefore, as appears from (2) and (3) water must diffuse from E_1 to E_2. The $O.W.A.$ of system E_2 (viz. of the system which has the smallest amount of water) is, therefore, greater than that of E_1.

We can apply those considerations also when one of the systems E_1 and E_2 or when both are osmotic invariant. Let us take f.i. the equilibria:

$$E_1 = H + Y + L_d \qquad E_2 = X + Y + L_c \quad \text{fig. 1 V.}$$

We imagine those equilibria to be represented by the points K_1 and K_2 (not drawn); consequently K_1 is situated within the triangle H Yd and K_2 within the triangle XYc of fig. 1 V. As E_2 passes into E_1 by taking in water, the points K_1 K_2 and W must be situated on a straight line. We now represent by K_3 the equilibrium E_3 which can be formed in accordance with reaction (1); this point K_3 is situated then anywhere on the line WK_1K_2 between K_1 and K_2; its place is defined by the ratio's of the quantities of E_1 and E_2 which we bring together. It now depends on the position of K_3 which equilibrium E_3 will arise. In accordance with the position of K_3 within the triangle HYd, the sector Ydc on the triangle XYc, E_3 is one of the equilibria:

$$H + Y + L_d \qquad Y + L \qquad X + Y + L_c \quad \text{fig. 1 V.}$$

The first of those equilibria is the same as E_1, with this difference, however, that the quantities of the phases are otherwise. The same is true for the last one, which is equal to E_2.

In similar way as above we now find that in (4) water must diffuse from E_1 towards E_2, so that the $O.W.A.$ of E_2 is greater than that of E_1.

We can apply similar considerations also to systems, in which one or more phases are the same (f.i. system $Y + D + L_c$ and $Y + D + X$ fig. 3 V), to systems which differ infinitely little in composition, etc. We now find:

Of two different stable systems, which can be converted into one another by taking in or losing water, and which are not in equilibrium with one another, the system with the smallest amount of water has the greatest $O.W.A.$

If in a stable equilibrium a phases-reaction occurs at taking in or

losing water, then it is osmotic invariant; its *O.W.A.* viz. remains constant. If no phases-reaction occurs, then it is osmotic variant; its *O.W.A.* becomes greater at losing water, and smaller at taking water in.

We now take the equilibrium

$$Y + L_b \quad \text{fig. 1 III}$$

which is represented in the figure by a point of the line bY f.i. point g. If we take away a little water from this system, then the liquid changes its composition; it shifts, starting from b along the saturation-curve in the direction towards v. As the *O.W.A.* of the system increases at losing water, the *O.W.A.* in the point b of the saturation-curve increases, therefore, in the direction of the arrow. If we apply this to all systems $Y + L$, in which L represents each liquid of the saturation-curve, then it appears, which we have already deduced previously in another way, that the *O.W.A.* of the liquids of the saturation-curve increases in the direction of the arrows, viz. from w to v.

We find the same for the liquids of the saturation-curve in fig. 2 III; here also the *O.W.A.* of the liquids increases in the direction from w towards v, notwithstanding the content of water of the liquids themselves firstly increases starting from w, in order to decrease later of course. It also appears in fig. 3 that the *O.W.A.* increases in the direction of the arrows.

We now take the equilibria

$$H + L \quad \text{fig. 1 IV} \qquad D + L \quad \text{fig. 2 IV}$$

in which L represents an arbitrary liquid of the saturation-curve. In the same way as above we now find the property, already previously deduced, that the *O.W.A.* increases from w to v.

In order to formulate this property in another way than formerly, we shall define the direction in which we move along a saturation-curve, in a corresponding way as in communication XIII for a binodalcurve. The line Yb (figs 1, 2 and 3 III) or Hb (fig. 1 IV) or Db (fig. 2 IV) divides the plane into two parts; in one of those parts the point W is situated. We now shall say that we go starting from b along the saturation-curve towards W, if we go along the saturation-curve towards that part of the plane, in which point W is situated. If we go in the other direction, then we shall say that we move away from the point W. We may summarize then the deduced above in the following way:

the *O.W.A.* of the liquids of a saturation-curve increases in that direction, in which we move away along this curve from the point W.

Of course this rule is valid also, if the diffusing substance is other than water. If f.i. the substance Y diffuses, then we have to replace in the above rule *O.W.A.* by *O.Y.A.* and point W by point Y.

Let us take the figures of the communications XI and XII in which Y is the diffusing substance. We then find in accordance with previously

that the $O.W.A.$ along the saturation curve $a\,d\,g$ of the substance D (fig. 2 XI) must increase in the direction of the arrows. The same is true for the saturation-curve $a\,b$ of the substance H, for the saturation-curve $c\,d$ of the substance X and for the saturation-curve $f\,g$ of *Ice* in fig. 2 XII.

We can apply also the rule on the change of the $O.W.A.$ of an osmotic variant equilibrium to define the direction, in which the $O.W.A.$ of the liquids of a binodal-curve increases. Let us take f.i. the equilibrium:

$$L_{a_1} + L_{a_2} \quad \text{fig. 1—3 XIII.}$$

If we take away from this system a little water, then those liquids shift along the binodal-curve away from the point W; consequently they move in the direction of the arrows. Those arrows, therefore, indicate also the direction, in which the $O.W.A.$ increases. In accordance with previously we find, therefore:

the $O.W.A.$ of the liquids of a binodal-curve increases in that direction in which we move away along this curve from the point W.

Consequently this rule is the same as that for the saturation-curve of a solid substance; the only difference is, that with the saturation-curve the conjugation-line solid-liquid, and with the binodal-curve the conjugation-line of two coexisting liquids plays a part.

We now take the equilibrium

$$E = L_{q_1} + L_{q_2} + L_{q_3} \quad \text{fig. 1 XV} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

When taking in water, herein occurs, as appears from the figure, the reaction:

$$L_{q_3} + water \rightarrow L_{q_1} + L_{q_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The quantity of the liquid q_3 decreases, therefore, that of the liquids q_1 and q_2 increases; with sufficient taking water in arises, therefore, the equilibrium $L_{q_1} + L_{q_2}$, which passes into an equilibrium $L(q_1 b_1) + L(q_2 b_2)$ by taking further water in. Herein $L(q_1\, b_1)$ represents a liquid of curve $q_1\, b_1$ and $L(q_2\, b_2)$ a liquid of curve $q_2\, b_2$. As this equilibrium arises from (5) by taking water in, it has, therefore, a smaller $O.W.A.$ than system (5). This is in accordance with the direction of the arrows on the binodal-curve $q_1\, a\, q_2$.

At losing water in the system (5) occurs the reaction:

$$L_{q_1} + L_{q_2} - water \rightarrow L_{q_3}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

with sufficient loss of water arises, therefore, one of the systems: $L_{q_1} + L_{q_3}$ or $L_{q_2} + L_{q_3}$. It depends on the ratio of the quantities of L_{q_1} and L_{q_2} in (5) which of those systems arises in a special case. With further loss of water arises one of the systems: $L(q_1\, a_1) + L(q_3\, a_3)$ or $L(q_2\, c_2) + L(q_3\, c_3)$ in which are indicated again between parentheses the curves on which the liquids are situated. As those equilibria arise from

(5) by losing water, they have a greater $O.W.A.$ than system (5); this is in accordance with the direction of the arrows in the figure.

We shall summarize the previous in the following way:

$$\left.\begin{array}{c} E = L_{q_1} + L_{q_2} + L_{q_3} \quad \text{fig. 1 XV} \\ L_{q_3} \rightleftarrows L_{q_1} + L_{q_2} \quad \Delta W > 0 \\ \begin{array}{l|l} L(q_3\, a_3) + L(q_1\, a_1) \\ L(q_3\, c_3) + L(q_2\, c_2) \end{array} \quad L(q_1\, b_1) + L(q_2\, b_2) \\ \longleftarrow O.W.A. \end{array}\right\} \quad \ldots \quad (8)$$

In the first line of (8) we find the osmotic invariant system, in the second the reaction which occurs between the phases at taking in and losing water. On the right side of this reaction we still find $\Delta W > 0$; this means that the reaction takes place from left to right with taking in water. In accordance with this we find at the right side of the vertical line the equilibrium, which is formed with taking in water, at the left side of this line the equilibria which are formed with loss of water. Consequently the equilibria at the left side of this line have a larger $O.W.A.$ than the equilibrium E, those at the right side of the line have a smaller $O.W.A.$ This is indicated by the direction of the lowest arrow.

We shall represent by (L_{q_3}) the equilibrium $L(q_1 b_1) + L(q_2\, b_2)$, which is formed from E, when herein disappears the phase L_{q_3}; the equilibrium $L(q_3 a_3) + L(q_1\, a_1)$ which is formed from E, when herein the phase L_{q_2} disappears, is represented by (L_{q_2}) and the equilibrium $L(q_3 c_3) + L(q_2 c_2)$ which is formed from E, when herein the phase L_{q_1} disappears, by (L_{q_1}). Instead of (8) we may write then:

$$\left.\begin{array}{c} E = L_{q_1} + L_{q_2} + L_{q_3} \quad \text{fig. 1 XV} \\ L_{q_3} \rightleftarrows L_{q_1} + L_{q_2} \quad \Delta W > 0 \\ (L_{q_1});\ (L_{q_2}) \mid (L_{q_3}) \\ \longleftarrow O.W.A. \end{array}\right\} \quad \ldots \quad (9)$$

For the equilibrium of the three liquids in fig. 2. XV is the reaction:

$$L_{q_1} + L_{q_2} \rightleftarrows L_{q_3} \qquad \Delta W > 0.$$

In (9) we have, therefore, to replace by one another the systems at the right and at the left of the vertical line. This is also in accordance with the direction of the arrows in this figure.

For the threephases-equilibrium of fig. 3. XV is found:

$$\left.\begin{array}{c} E = F + L_{q_1} + L_{q_2} + \quad \text{fig. 3 XV} \\ F + L_{q_2} \rightleftarrows L_{q_1} \quad \Delta W > 0 \\ (L_{q_1}) \mid (F);\ (L_{q_2}) \\ \longleftarrow O.W.A. \end{array}\right\} \quad \ldots \quad (10)$$

Herein (L_{q_1}) represents the equilibrium, which arises from E, if the

liquid L_{q_1} disappears from this; consequently we have: $(L_{q_1}) = F + L$, in which L is a liquid of curve $q_2 f$. In accordance with (10) the $O.W.A.$ of this system must be larger than that of equilibrium E; this is in accordance with the direction of the arrow on curve $d_2 f$ of the figure.

(F) represents the system, which arises from E if the substance F disappears from this, consequently two conjugated liquids of the binodal-curve $q_1 \alpha q_2$; in accordance with (10) the $O.W.A.$ of this system is smaller than that of the equilibrium E; the same is true for the equilibrium $(L_{q_2}) = F + L$, in which L represents a liquid of curve $q_1 w$.

In the equilibrium $D + X + L_d$ of fig. 3. V occurs the reaction:

$$D + X \rightleftarrows L_d \qquad \Delta W > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

We may write, therefore:

$$\left.\begin{array}{l} E = D + X + L_d \quad \text{fig. 3 V} \\ D + X \rightleftarrows L_d \quad \Delta W > 0 \\ (L_d) \mid (D);\ (X) \\ \longleftarrow O.W.A. \end{array}\right\} \quad . \quad . \quad . \quad (12)$$

The equilibria $(D) = X + L$ and $(X) = D + L$ have, therefore, a smaller $O.W.A.$ than the equilibrium E; the direction of the arrows on the curves db and ds is in accordance with this. As (L_d) represents the equilibrium which is formed from E by loss of water, we may have, in accordance with the figure: $(L_d) = D + X + Y$. The $O.W.A.$ of this system is larger, therefore, than that of the system E. We have already deduced this in another way in communication VI.

We now take the same threephases-equilibrium as in (12), but now in fig. 4. V. We then find:

$$\left.\begin{array}{l} E = D + X + L_d \quad \text{fig. 4 V} \\ D \longrightarrow X + L_d \quad \Delta W > 0 \\ (X);\ (L_d) \mid (D) \\ \longleftarrow O.W.A. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (13)$$

The equilibrium $(X) = D + L$ is situated now on the other side of the vertical line than in (12); therefore, the $O.W.A.$ must increase along the saturation-curve of D in fig. 4. V starting from the point d.

In the equilibrium $E = H + Y + L_d$ of fig. 2. V occurs the reaction:

$$Y \rightleftarrows H \qquad \Delta W > 0.$$

As the liquid does not take part in the reaction we may call E a singular equilibrium [1]) and L_d its indifferent phase. We now fiind:

$$\left.\begin{array}{l} E = H + Y + L_d \quad \text{fig. 2 V} \\ Y \rightleftarrows H \quad \Delta W > 0 \\ (H) \mid (Y) \\ \longleftarrow O.W.A. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (14)$$

[1]) F. A. H. SCHREINEMAKERS, In-, mono- and plurivariant equilibria XXVII and XXVIII.

In accordance with the arrows in fig. 2, V it follows from this that $(H)=Y+L$ has a greater and $(Y)=H+L$ has a smaller $O.W.A.$ than the system E.

If we introduce a semi-permeable membrane in an equilibrium E then it does not change its stability, we then get an osmotic equilibrium:

$$E_1 \,|\, E_2 \quad \ldots\ldots\ldots \quad (15)$$

If we represent the thermodynamical potential of the system E by Z, then, if this system contains one or more phases, which change their composition at taking in or losing water, and if E_1 and E_2 each contain the half of the phases of system E, at diffusion of δn quantities of water the thermodynamical potential at the one side of the membrane will become:

$$\frac{1}{2}\left[Z+\frac{\partial Z}{\partial n}.\delta n+\frac{1}{2}.\frac{\partial^2 Z}{\partial n^2}.\partial n^2+\ldots\right]$$

and at the other side:

$$\frac{1}{2}\left[Z-\frac{\partial Z}{\partial n}.\delta n+\frac{1}{2}.\frac{\partial^2 Z}{\partial n^2}.\partial n^2-\ldots\right].$$

Consequently the total thermodynamical potential becomes:

$$Z+\frac{1}{2}\frac{\partial^2 Z}{\partial n^2}.\partial n^2+\ldots, \quad \ldots\ldots \quad (16)$$

If we assume that system E and, therefore, also the osmotic system (15) is stable for small disturbances of equilibrium, then (16) must be greater than Z, so that $\frac{\partial^2 Z}{\partial n^2}$ is positive. As $\frac{\partial Z}{\partial n}=\varphi, \frac{\partial \varphi}{\partial n}$ must be positive, therefore. Consequently the φ of system E increases at taking in a little water. Hence follows the rule, already deduced above:

the $O.W.A.$ of a stable osmotic variant equilibrium becomes greater when losing water and smaller when taking water in.

This rule needs however to be true no more for systems, which are unstable for small changes of equilibrium. Let us imagine in a diagram f. i. in fig. 1 XIV a straight line to be drawn through point W. We take on this line a point Q and we represent the composition of a liquid L by:

x quant. of $W+y$ quant. of $N+(1-x-y)$ quant. of Q

in which N is an arbitrary phase. The $O.W.A.$ of this liquid L is defined by:

$$\varphi=\zeta+(1-x)\frac{\partial \zeta}{\partial x}-y\frac{\partial \zeta}{\partial y} \quad \ldots\ldots \quad (17)$$

For a new liquid L', which differs infinitely little from L, is true, therefore:

$$d\varphi=[(1-x)\,r-ys]\,dx+[(1-x)\,s-yt]\,dy \quad \ldots \quad (18)$$

If we imagine the liquid L to be in the point Q and the new liquid L' on the line WQ (so that $dy = 0$) then (18) passes into:

$$d\varphi = r\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Wo now go in fig. 1 XIV f.i. along the line Wb from the liquid L towards the new liquid L', which is situated somewhat further from point W. As then dx is negative and r is positive in all points of this line $d\varphi$ is negative, therefore. The same is true of course for all liquids of the parts of a straight line going through point W, which are situated outside the region of dimixtion. Consequently we find in accordance with previously:

the $O.W.A.$ of a stable liquid becomes greater when losing water and smaller when taking water in.

Otherwise it is, however, if we take a liquid which is situated within the spinodalcurve (and, therefore, surely within the region of dimixtion); then r can be negative. If we take f.i. the line Wu of fig. 1 XIV; previously we have assumed that r in the points c and m is zero, and negative between these points. In connection with this an isotonic curve touches this line Wu in c and an other curve (not drawn) touches it in m. It now follows from (19):

the $O.W.A.$ of a liquid between c and m (fig. 1 XIV) becomes smaller when losing water and greater when taking water in.

We see this also from the figure, if we imagine to be drawn in this still some isotonic curves, situated between the two, which touch in c and m the line Wu.

Also the rule, deduced above, that the $O.W.A.$ of the liquids of a saturation-curve increases in that direction in which we move along this curve from the point W, needs to be true no more when the system is unstable for small changes in equilibrium. Let us take f.i. the saturation-curve $a\,q_1\,q_2\,f$ of the substance F in fig. 3 XV. It appears from the direction of the arrows that the rule above mentioned is not valid for the parts $l\,m$ and $p\,n$ of this curve, which are situated within the region of dimixtion, but it is valid for the parts $q_1\,l$, $m\,n$ and $p\,q_2$ which are situated also within the region of dimixtion.

(To be continued.)

Mathematics. — *"Intuitionistische Einführung des Dimensionsbegriffes."* By Prof. L. E. J. BROUWER.

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§ 1. *Definition der katalogisiertkompakten Spezies.*

Unter einer *Teilmenge der Menge M* verstehen wir im folgenden eine derweise aus M entstehende Menge, dass mittels einer unbeschränkten Folge von Entscheidungen der Reihe nach für jede ungehemmte endliche Wahlfolge von M entschieden wird, entweder dass sie ungehemmt bleibt, oder dass sie gehemmt wird. Der Mengenbegriff erleidet hierbei gegenüber der Math. Ann. 93, S. 244 gegebenen Definition insofern eine Erweiterung, dass die Entscheidung zwischen Gehemmtheit und Ungehemmtheit nicht mehr von vornherein für alle endlichen Wahlfolgen durch ein Gesetz gänzlich festgelegt zu sein braucht.

Unter einer *katalogisierten Folge* verstehen wir eine unbegrenzte Folge $p_1, p_2, \ldots$, in welcher zu je zwei Elementen p_{ν_1} und p_{ν_2} ein Abstand $\varrho(p_{\nu_1}, p_{\nu_2}) \geqq 0$ (aber nicht notwendig entweder > 0 oder $= 0$) definiert wird, der folgenden Bedingungen genügt: Es ist stets $\varrho(p_\nu, p_\nu) = 0$ und $\varrho(p_{\nu_1}, p_{\nu_2}) \leqq \varrho(p_{\nu_1}, p_{\nu_3}) + \varrho(p_{\nu_2}, p_{\nu_3})$. Weiter existiert zu jedem n ein μ_n und ein ν_n, so dass für $\nu > \nu_n$ jedes p_ν in einer Entfernung $\leqq a <_\circ 4^{-n}$ von $s_{\mu_n} = \mathfrak{S}(p_1, p_2, \ldots, p_{\mu_n})$ gelegen ist. Schliesslich ist jedes Element p_{τ_n} von s_{μ_n} entweder ein *β_n-Element* oder ein *α_n-Element* (niemals aber beides gleichzeitig); im ersteren Falle ist für $\nu > m(\eta_n)$ jedes p_ν in einer Entfernung $\geqq \frac{5}{4} \cdot 4^{-n}$ von $p_{\nu_{\eta_n}}$ gelegen, im letzteren Falle gibt es eine unbegrenzte Folge von verschiedenen p_ν in einer Entfernung $\leqq b <_\circ \frac{3}{2} \cdot 4^{-n} \left(b \circ > \frac{5}{4} \cdot 4^{-n} \right)$ von p_{τ_n}. (Demzufolge besitzt dann auch ein beliebiges p_ν entweder die für die α_n oder die für die β_n charakteristische Eigenschaft.)

Was wir auf dieser Grundlage unter einer *positiv-konvergenten unbegrenzten Folge* (bzw. unter *zusammengehörigen positiv-konvergenten unbegrenzten Folgen*) von verschiedenen Elementen einer katalogisierten Folge zu verstehen haben, ist ohne weiteres klar.

Die Spezies S der *Spezies zusammengehöriger positiv-konvergenter unbegrenzter Folgen von verschiedenen Elementen* einer katalogisierten Folge F ist *kompakt*, d.h. wenn wir in auf der Hand liegender Weise die Entfernung je zweier Elemente von S definieren, so konvergiert jede

positiv-konvergente Folge von Elementen von S positiv gegen ein Element von S. Weiter ist S *in bezug auf* F *katalogisiert*, d.h. die Entfernung zwischen S und einem beliebigen Elemente p_ν von F lässt sich mit jedem beliebigen Grade der Genauigkeit approximieren (ohne dass jedoch stets ein Element q von S angegeben werden kann so dass $\varrho(q, p_\nu) = \varrho(S, p_\nu)$). Die Spezies S *„fällt zusammen"* mit einer kompakten finiten Menge M von positiv-konvergenten unbegrenzten Folgen von (eventuell teilweise identischen) Elementen von F, welche erhalten werden, indem in solcher Weise der Reihe nach ein α_1-Element, ein α_2-Element, u.s.w. gewählt werden, dass nach der Wahl eines bestimmten α_n-Elementes α_n^0 jedes α_{n+1}-Element in einer Entfernung $\leqslant 2.4^{-n}$ von α_n^0 wählbar und jedes α_{n+1}-Element in einer Entfernung $\geqslant 3.4^{-n}$ von α_n^0 unwählbar ist. Die Folge F_1, welche der Reihe nach die α_1-Elemente, die α_2-Elemente, u.s.w. enthält, ist wiederum katalogisiert (wobei man diesmal ohne die Analoga der β_n-Elemente auskommen kann) und die Spezies der Spezies zusammengehöriger positiv-konvergenter unbegrenzter Folgen von verschiedenen Elementen von F_1 fällt zusammen mit S. Jede in derselben Weise wie S mittels einer katalogisierten Folge definierte Spezies, und überdies die leere Spezies, heisst eine *katalogisiert-kompakte Spezies*.

Sei S' eine in bezug auf F_1 katalogisierte kompakte Teilspezies von S (also z.B. die „Abschliessung" einer beliebigen Teilmenge von M; übrigens werden wir auch die leere Spezies zu den in bezug auf F_1 katalogisierten kompakten Teilspezies von S rechnen). Jedem α_n-Elemente ${}_1\alpha_n$ geben wir entweder das Prädikat eines *α_n'-Elementes* (in welchem Falle es Elemente von S' in einer Entfernung $\leqslant c <_\circ \frac{3}{2} \cdot 4^{-n}$ $(c >_\circ \frac{5}{4} \cdot 4^{-n})$ von ${}_1\alpha_n$ gibt) oder das Prädikat eines *β_n'-Elementes* (in welchem Falle jedes Element von S' in einer Entfernung $\geqslant \frac{5}{4} \cdot 4^{-n}$ von ${}_1\alpha_n$ gelegen ist), mit der Massgabe, dass die beiden Prädikate sich gegenseitig ausschliessen. Sodann bestimmen wir eine mit S' zusammenfallende Teilmenge M' von M, für welche in solcher Weise der Reihe nach ein α_1'-Element, ein α_2'-Element, u. s. w. gewählt werden, dass nach der Wahl eines bestimmten α_n'-Elementes ${}_0\alpha_n'$ jedes nach derselben für M unwählbare α_{n+1}'-Element sowie jedes in einer Entfernung $\geqslant 3.4^{-n}$ von ${}_0\alpha_n'$ gelegene α_{n+1}'-Element unwählbar und jedes in einer Entfernung $\leqslant 2.4^{-n}$ von ${}_0\alpha_n'$ gelegene α_{n+1}'-Element wählbar ist. Die Folge F_1', welche der Reihe nach die α_1'-Elemente, die α_2'-Elemente, u.s.w. enthält, ist wiederum katalogisiert (wobei man wiederum ohne die Analoga der β_n-Elemente auskommen kann) und die Spezies der Spezies zusammengehöriger positiv-konvergenter unbegrenzter Folgen von verschiedenen Elementen von F_1' fällt zusammen mit S'.

In analoger Weise stellt sich heraus, dass eine in bezug auf F_1' katalogisierte kompakte Teilspezies S'' von S' mit einer Teilmenge M'' von

M' zusammenfällt; demzufolge ist sie ebenfalls in bezug auf F_1 (sowie in bezug auf F und in bezug auf S) katalogisiert.

§ 2. *Ein Satz über Erweiterung von Trennungen in katalogisiertkompakten Spezies.*

In Anschluss an eine vorgegebene in bezug auf F_1 katalogisierte kompakte Teilspezies S' von S bestimmen wir in folgender Weise eine Teilmenge $M_n'(S')$ von M: Jedem α_{n+2}-Elemente ${}_1\alpha_{n+2}$ sprechen wir entweder das Prädikat eines α_{n+2}''-*Elementes* zu (in welchen Falle es in einer Entfernung $\geqq 4^{-n}+4^{-n-1}$ von S' gelegen ist) oder das Prädikat eines β_{n+2}''-*Elementes* (in welchem Falle es einen Abstand $\leqq 4^{-n}+2.4^{-n-1}$ von S' besitzt). Alsdann wird $M_n'(S')$ von denjenigen zu M gehörigen Folgen, deren $(n+2)$-tes Element ein α_{n+2}''-Element ist, gebildet. Jedes Element der Abschliessung $A_n(S')$ von $M_n'(S')$ besitzt einen Abstand $\geqq 4^{-n}$ von S', während jedes in einer Entfernung $\geqq 4^{-n}+2.4^{-n-1}+4^{-n-2}$, a fortiori also jedes in einer Entfernung $\geqq 2.4^{-n}$ von S' gelegene Element von S zu $A_n(S')$ gehört. Sei $M_n(S')$ eine nach dem Verfahren des § 1 konstruierte mit $A_n(S')$ zusammenfallende Teilmenge von M, und sei jeder Teilmenge τ von $M_n(S')$ eine positive rationale Zahl ϱ_τ zugeordnet. Alsdann bestehen, *weil die Spezies der τ eine finite Menge darstellt*, nach dem in diesen Proceedings XXVII, S. 192 (vgl. auch S. 646) hergeleiteten Satze, für ϱ_τ nur endlichviele Werte und unter denselben ein positiver, rationaler Minimalwert.

Wir sagen, dass die Teilspezies α von S zwischen den Teilspezies β und γ von S eine *Sprungweite* b aufweist, wenn in jeder endlichen Folge von Elementen von S, von denen das erste zu β, das letzte zu γ und alle anderen zu α gehören, zwei aufeinanderfolgende Elemente vorkommen, deren Abstand $\geqq b$ ist. Im Falle, dass α, β und γ kompakt und in bezug auf F_1 katalogisiert sind, besteht offenbar eine Maximalsprungweite von α zwischen β und γ, welche sich mit jedem beliebigen Grade der Genauigkeit approximieren lässt.

Seien A und B in bezug auf F_1' katalogisierte kompakte Teilspezies von S' und C und D in bezug auf F_1 katalogisierte kompakte Teilspezies von S, während von den Spezies A, B und C je zwei einen Abstand $\circ > 0$ besitzen, und seien A und B in S' durch C *getrennt*, womit wir (für Spezies S, S', A, B und C der angegebenen Art) meinen, dass jede in bezug auf F_1' katalogisierte kompakte Teilspezies S'' von S', welche eine Entfernung $\circ > 0$ von C besitzt, eine Sprungweite (mithin auch eine rationale Sprungweite) $\circ > 0$ zwischen A und B aufweist[1]). Alsdann

[1]) Dieser *"schwachen Trennung"* lassen sich an die Seite stellen:

1. Die *„starke Trennung"*, welche stattfindet, wenn jedes in einer Entfernung $\circ > 0$ von C gelegene Element von S' entweder mit dem Prädikate eines g_A-Elementes, oder mit

können wir nach der obigen Methode eine mit einer in bezug auf F_1' katalogisierten kompakten Teilspezies von S' zusammenfallende Teilmenge ${}_{S'}M_n(C)$ von M' bestimmen, so dass jedes Element von ${}_{S'}M_n(C)$ einen Abstand $\geqq 4^{-n}$ von C besitzt, während jedes in einer Entfernung $\geqq 2.4^{-n}$ von C liegende Element von S' mit einem Elemente von ${}_{S'}M_n(C)$ zusammenfällt. Jeder Teilmenge von ${}_{S'}M_n(C)$ ist als Sprungweite ihrer Abschliessung zwischen A und B eine positive rationale Zahl zugeordnet; diese positive rationale Zahl besitzt nach dem ersten Absatze dieses § einen Minimalwert σ_n, welcher als Sprungweite der Abschliessung einer *beliebigen* Teilmenge von ${}_{S'}M_n(C)$ zwischen A und B, mithin insbesondere als Sprungweite einer *beliebigen* in einer Entfernung $\geqq 2.4^{-n}$ von C gelegenen, in bezug auf F_1' katalogisierten kompakten Teilspezies von S' zwischen A und B auftritt. Wenn n hinreichend gross gewählt wird, dürfen wir, wenn wie immer $\varrho(X, Y)$ den Abstand zwischen X und Y vorstellt, annehmen, dass σ_n, sowie 2.4^{-n+1}, $\leqq \varrho(A, B)$, $\leqq \varrho(A, C)$ und $\leqq \varrho(B, C)$ sind.

Sei A_1 (bzw. B_1) eine solche gänzlich in einem Abstande $<\circ \frac{1}{4}\sigma_n$ von S' gelegene, in bezug auf F_1 katalogisierte kompakte Teilspezies von S, dass jedes in einer Entfernung $\leqq \frac{1}{4}\sigma_n$ von A_1 (bzw. von B_1) gelegene Element von S' zu A (bzw. zu B) gehört, so dass A_1 (bzw. B_1) auch gänzlich in einem Abstande $<\circ \frac{1}{4}\sigma_n$ von A (bzw. von B) gelegen ist. Sei T eine gänzlich in einer Entfernung $<\circ \frac{1}{4}\sigma_n$ von S' und gänzlich in einer Entfernung $\geqq 2.4^{-n} + \frac{1}{4}\sigma_n$ von C gelegene, in bezug auf F_1 katalogisierte kompakte Teilspezies von S. *Alsdann weist T zwischen A_1 und B_1 eine Sprungweite $\frac{1}{2}\sigma_n$ auf.* Wenn wir nämlich eine endliche Folge von Elementen von S betrachten, von denen das erste zu A_1, das letzte zu B_1 und alle anderen zu T gehören, so können wir, indem

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dem Prädikate eines g_B-Elementes (welche Prädikate sich gegenseitig ausschliessen) versehen ist, mit der Massgabe, dass jedes Element von A (bzw. von B) ein g_A-Element (bzw. ein g_B-Element) ist, und dass zu jedem $\eta_1 \circ > 0$ ein solches $\eta_2 \circ > 0$ bestimmt werden kann, dass jedes in einer Entfernung $\leqq \eta_2$ von einem in einer Entfernung $\geqq \eta_1$ von C gelegenen g_A-Elemente (bzw. g_B-Elemente) liegende Element von S' ebenfalls ein g_A-Element (bzw. g_B-Element) ist.

2. Die "*mittlere Trennung*", welche stattfindet, wenn jedes in einer Entfernung $\circ > 0$ von C gelegene Element von S' entweder die t_A-Eigenschaft oder die t_B-Eigenschaft (welche sich übrigens nicht ausschliessen) besitzt. Dabei sagen wir, dass ein Element q' von S' die t_A-Eigenschaft (bzw. die t_B-Eigenschaft) besitzt, wenn ein $\eta(q') \circ > 0$ existiert, so dass jede in bezug auf F_1' katalogisierte kompakte Teilspezies von S', welche gänzlich in einer Entfernung $\leqq \eta(q')$ von q' gelegen ist, in S' durch C von A (bzw. von B) "schwach getrennt" wird.

wir jedes Element der Folge einer passenden Verrückung $<\circ \frac{1}{4}\sigma_n$ unterziehen, eine endliche Folge von Elementen von S erzeugen, von denen das erste zu A, das letzte zu B und alle anderen zu S' gehören und in einer Entfernung $\circ> 2.4^{-n}$ von C liegen, in welcher mithin zwei aufeinanderfolgende Elemente vorkommen, deren Abstand $\geqq \sigma_n$ ist.

Sei R eine alle Elemente von D in einem Abstande $\leqq \frac{1}{8}\sigma_n$ von S' enthaltende und gänzlich in einer Entfernung $<\circ \frac{1}{4}\sigma_n$ von S' gelegene, in bezug auf $\mathfrak{F}_1$ katalogisierte kompakte Teilspezies von D; seien A_1 und B_1 den im vorigen Absatz aufgestellten Forderungen entsprechend, und überdies innerhalb R gewählt; und sei C_1 eine alle Elemente von D in einem Abstande $\leqq 2.4^{-n} + \frac{1}{4}\sigma_n$ von C enthaltende, in einem Abstande $\circ> \frac{1}{4}\sigma_n$ von A sowie von B gelegene, in bezug auf $\mathfrak{F}_1$ katalogisierte kompakte Teilspezies von D. Alsdann besitzen erstens je zwei der Spezies A_1, B_1 und C_1 einen Abstand $\circ> 0$ voneinander, zweitens sind nach dem vorigen Absatze *A_1 und B_1 in R durch C_1 getrennt.*

Mit dem obigen ist folgende Eigenschaft bewiesen: *Sei S eine katalogisiertkompakte Spezies; S' und D in bezug auf S katalogisierte kompakte Teilspezies von S; A und B je eine Relativumgebung von S' enthaltende, in einem Abstande $\circ> 0$ voneinander gelegene, in bezug auf S katalogisierte kompakte Teilspezies von S'; C eine in einem Abstande $\circ> 0$ von A sowohl wie von B gelegene, A und B in S' trennende, in bezug auf S katalogisierte kompakte Teilspezies von S; C_1 eine alle Elemente von D in einem Abstande $\leqq \alpha_1$ von C enthaltende, in einem Abstande $\circ> \alpha_2 \circ> 0$ von A sowohl wie von B gelegene, in bezug auf S katalogisierte kompakte Teilspezies von D; n und ein zugehöriges $\sigma(n)$ (d.h. eine für eine beliebige in einer Entfernung $\geqq 2.4^{-n}$ von C gelegene, in bezug auf S katalogisierte kompakte Teilspezies von S' bestehende Sprungweite zwischen A und B) so gewählt, dass je zwei der Spezies A, B und C in einer Entfernung $\geqq \sigma_n$ und $\geqq 2.4^{-n+1}$ voneinander gelegen sind und überdies die Ungleichungen $2.4^{-n} + \frac{1}{4}\sigma_n \leqq \alpha_1$ und $\frac{1}{4}\sigma_n <\circ \alpha_2$ bestehen; R eine alle Elemente von D in einem Abstande $\leqq \frac{1}{8}\sigma_n$ von S' enthaltende, gänzlich innerhalb einer Entfernung $<\circ \frac{1}{4}\sigma_n$ von S' gelegene, in bezug auf S katalogisierte kompakte Teilspezies von D; A_1 (bzw. B_1) eine solche in bezug auf S katalogisierte kompakte Teilspezies von R, dass jedes in einer Entfernung $\leqq \frac{1}{4}\sigma_n$ von A_1 (bzw. B_1) gelegene Element von S' zu A (bzw. B) gehört. Alsdann sind A_1 und B_1 in R durch C_1 getrennt.*

§ 3. *Der Begriff des Dimensionsgrades für katalogisiertkompakte Spezies.*

Wir sagen, dass die kompaktkatalogisierte Spezies S den *unteren Dimensionsgrad* n besitzt, wenn in S zwei in bezug auf S katalogisierte kompakte Teilspezies A und B angegeben werden können, welche in einem Abstande $\circ > 0$ voneinander gelegen sind und die Eigenschaft besitzen, dass jede A und B in S trennende [2]), in bezug auf S katalogisierte kompakte Teilspezies von S den unteren Dimensionsgrad $n-1$ besitzt. Weiter sagen wir, dass S den *unteren Dimensionsgrad Null bzw.* ω besitzt, wenn ein Element von S angegeben werden kann, bzw. wenn S für jede natürliche Zahl n den unteren Dimensionsgrad n besitzt.

Wir sagen, dass die kompaktkatalogisierte Spezies S den *oberen Dimensionsgrad* n besitzt, wenn je zwei in einem Abstande $\circ > 0$ voneinander gelegene, in bezug auf S katalogisierte kompakte Teilspezies von S durch eine den oberen Dimensionsgrad $n-1$ besitzende, in bezug auf S katalogisierte kompakte Teilspezies von S, in S getrennt werden können. Weiter sagen wir, dass S den *oberen Dimensionsgrad Null bzw.* ω besitzt, wenn S zwischen je zwei in einem Abstande $\circ > 0$ voneinander gelegenen, in bezug auf S katalogisierten kompakten Teilspezies von S eine Sprungweite $\circ > 0$ aufweist bzw. wenn je zwei in einem Abstande $\circ > 0$ voneinander gelegene, in bezug auf S katalogisierte kompakte Teilspezies von S durch eine einen endlichen oberen Dimensionsgrad besitzende, in bezug auf S katalogisierte kompakte Teilspezies von S, in S getrennt werden können.

Wenn eine kompaktkatalogisierte Spezies sowohl den oberen wie den unteren Dimensionsgrad n bzw. 0 bzw. ω besitzt, so sagen wir, dass sie den *allgemeinen Dimensionsgrad* n *bzw.* 0 *bzw.* ω besitzt.

Den obigen Definitionen lässt sich leicht eine von der Rekurrenz unabhängige Form geben. Dazu denken wir uns die Spezies S von zwei Personen P und Q der „*Dimensionsoperation*" unterzogen, worunter wir folgendes verstehen: P wählt in S zwei in einem Abstande $\circ > 0$ voneinander gelegene, in bezug auf S katalogisierte kompakte Teilspezies A und B, worauf Q die Spezies A und B in S trennt durch eine in bezug auf S katalogisierte kompakte Teilspezies S_1 von S. Sodann wählt P zwei in einem Abstande $\circ > 0$ voneinander gelegene, in bezug auf S_1 katalogisierte kompakte Teilspezies A_1 und B_1 von S_1, worauf Q die Spezies A_1 und B_1 in S_1 trennt durch eine in bezug auf S_1 katalogisierte kompakte Teilspezies S_2 von S_1. Dieser Prozess wird unbeschränkt wiederholt, bis eventuell eine Spezies S_h auftritt, welche zwischen je zwei in einem Abstande $\circ > 0$ voneinander gelegene, in bezug auf S_h katalogisierten kompakten Teilspezies von S_h eine Sprungweite $\circ > 0$ aufweist.

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[2]) Zu je zwei in einem Abstande $\circ > 0$ voneinander gelegenen, in bezug auf S katalogisierten kompakten Teilspezies von S kann *immer eine* in einem Abstande $\circ > 0$ sowohl von A wie von B gelegene, in bezug auf S katalogisierte kompakte Teilspezies von S angegeben werden, welche A und B in S trennt.

Wenn nun Q unabhängig von den Wahlen der A_ν und B_ν (inklusive A und B) dafür sorgen kann, dass eine Spezies S_h auftritt, deren $h \leqq n$, so sagen wir, dass S den oberen Dimensionsgrad n besitzt. Wenn dagegen P unabhängig von den Wahlen der S_ν dafür sorgen kann, dass keine Spezies S_h auftritt, deren $h < n$, so sagen wir, dass S den unteren Dimensionsgrad n besitzt[3]).

§ 4. *Die Dimensionsgrade der n-dimensionalen Mannigfaltigkeiten.*

SATZ 1. *Jedes n-dimensionale Fragment S* (worunter wir hier die „vereinigende" katalogisiertkompakte Punktkernspezies der *n*-dimensionalen Elemente von *S* verstehen) *besitzt den oberen Dimensionsgrad n.*

BEWEIS. Nachdem *P* in *S* die Spezies *A* und *B* gewählt hat, bestimmt *Q* eine so dichte simpliziale Zerlegung $\mathfrak{Z}$ von *S*, dass er auf Grund derselben eine Gruppe *g* von Grundsimplexen von $\mathfrak{Z}$ und zwei Grössen $a \circ> 0$ und $b \circ> 0$ bestimmen kann mit der Eigenschaft, dass jedes in einem Abstande $\leqq a$ von *A* gelegene Grundsimplex von $\mathfrak{Z}$ zu *g* gehört und jedes zu *g* gehörige Grundsimplex von $\mathfrak{Z}$ einen Abstand $\geqq b$ von *B* besitzt. Die Gruppe *g* bildet ein *n*-dimensionales Fragment, deren *A und B in S trennende* Grenze, welche von *Q* als S_1 gewählt wird, aus der „vereinigenden" katalogisiertkompakten Puntkernspezies einer endlichen Anzahl von geschlossenen $(n-1)$-dimensionalen Pseudomannigfaltigkeiten besteht, mithin ein $(n-1)$-dimensionales Fragment bildet (in welchem übrigens mehrere Elementseiten, welche als verschieden zu betrachten sind, zusammenfallen können). Falls darauf *P* die Spezies A_1 und B_1 im selben Teilkontinuum von S_1 wählt, bestimmt *Q* eine so dichte simpliziale Zerlegung $\mathfrak{Z}_1$ von S_1, dass er auf Grund derselben eine Gruppe g_1 von Grundsimplexen von $\mathfrak{Z}_1$ und zwei Grössen $a_1 \circ> 0$ und $b_1 \circ> 0$ bestimmen kann mit der Eigenschaft dass jedes in einem Abstande $\leqq a_1$ von A_1 gelegene Grundsimplex von $\mathfrak{Z}_1$ zu g_1 gehört und jedes zu g_1 gehörige Grundsimplex von $\mathfrak{Z}_1$ einen Astand $\geqq b_1$ von B_1 besitzt. Die Gruppe g_1 bildet ein $(n-1)$-dimensionales Fragment, deren A_1 *und* B_1 *in* S_1 *trennende* Grenze, welche von *Q* als S_2 gewählt wird, aus der „vereinigenden" katalogisiertkompakten Punktkernspezies einer endlichen Anzahl von geschlossenen $(n-2)$-dimensionalen Pseudomannigfaltigkeiten besteht, mithin ein $(n-2)$-dimensionales Fragment bildet (in welchem übrigens wieder mehrere Elementseiten, welche als verschieden zu betrachten sind, zusammenfallen können). In dieser Weise fortfahrend, ge-

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[3]) Die im § 3 aufgestellten Definitionen bilden mit der im § 4 enthaltenen Beweisführung eine intuitionistische Ausarbeitung der in diesen Proceedings Bd. XXVII, S. 636—638 gegebenen Variante zur Einführung des *"natürlichen Dimensionsbegriffes"*. Ich benutze diese Gelegenheit, um darauf hinzuweisen, dass a. a. O. S. 636 Z. 13—14 anstatt: *jede zusammenhängende Teilspezies von π, welche sowohl mit ρ wie mit ρ' Punkte gemeinsam hat*, zu lesen ist: *jede zwischen ρ und ρ' zusammenhängende Teilspezies von π, welche sowohl mit ρ wie mit ρ' Punkte gemeinsam hat.*

langt Q schliesslich zu einem aus endlichvielen Punkten bestehenden S_n, es sei denn, dass der Prozess schon früher dadurch beendet wurde, dass P einmal A_ν und B_ν nicht in demselben Teilkontinuum von S_ν wählte.

SATZ 2. *Jede kompakte Teilspezies S eines n-dimensionalen Fragmentes F, welche in bezug auf F katalogisiert ist und ein Grundsimplex G einer simplizialen Zerlegung von F enthält, besitzt den unteren Dimensionsgrad n.*

BEWEIS. Seien $E_1, E_2, \ldots, E_{n+1}$ die Eckpunkte eines Grundsimplexes einer simplizialen Zerlegung von G, das von allen Seiten von G einen Abstand $\circ\!\!> 0$ besitzt. Alsdann wählt P für A (bzw. B) eine in bezug auf S katalogisierte kompakte Teilspezies von S, welche für gewisse Grössen ε und $\varepsilon' \circ\!\!> 0$ alle in einer Entfernung $\leqslant \varepsilon$ von E_1 (bzw. vom Simplex $E_2 E_3 \ldots E_{n+1}$) gelegenen Punktkerne von S enthält und eine Entfernung $\geqslant \varepsilon'$ vom Simplex $E_2 E_3 \ldots E_{n+1}$ (bzw. von E_1) besitzt, wobei dafür gesorgt wird, dass A und B in einer Entfernung $\circ\!\!> 0$ voneinander liegen; für A_1 (bzw. B_1) eine in bezug auf S_1 katalogisierte kompakte Teilspezies von S_1, welche für gewisse Grössen ε_1 und $\varepsilon_1' \circ\!\!> 0$ alle in einer Entfernung $\leqslant \varepsilon_1$ vom Segment $E_1 E_2$ (bzw. vom Simplex $E_1 E_3 \ldots E_{n+1}$) gelegenen Punktkerne von S_1 enthält und eine Entfernung $\geqslant \varepsilon_1'$ vom Simplex $E_1 E_3 \ldots E_{n+1}$ (bzw. vom Segment $E_1 E_2$) besitzt, wobei dafür gesorgt wird, dass A_1 und B_1 in einer Entfernung $\circ\!\!> 0$ voneinander liegen; für A_2 (bzw. B_2) eine in bezug auf S_2 katalogisierte kompakte Teilspezies von S_2, welche für gewisse Grössen ε_2 en $\varepsilon_2' \circ\!\!> 0$ alle in einer Entfernung $\leqslant \varepsilon_2$ vom Simplex $E_1 E_2 E_3$ (bzw. vom Simplex $E_1 E_2 E_4 \ldots E_{n+1}$) gelegenen Punktkerne van S_2 enthält und eine Entfernung $\geqslant \varepsilon_2'$ vom Simplex $E_1 E_2 E_4 \ldots E_{n+1}$ (bzw. vom Simplex $E_1 E_2 E_3$) besitzt, wobei dafür gesorgt wird, dass A_2 und B_2 in einer Entfernung $\circ\!\!> 0$ voneinander liegen; u. s. w. Um zu beweisen, dass von den Spezies $S_1, S_2, \ldots, S_n$ keine in Fortfall kommen kann, bezeichnen wir mit T das (abgeschlossene) Simplex $E_1 E_2 E_3 \ldots E_{n+1}$ und konstruieren nach irgend einem Gesetz eine Fundamentalreihe $\mathfrak{Z}_1, \mathfrak{Z}_2, \ldots$ von simplizialen Zerlegungen von T, deren jede eine Unterteilung der ihr vorangehenden ist, während für unbeschränkt wachsendes ν die Maximalbreite der Grundsimplexe von $\mathfrak{Z}_\nu$ positiv gegen Null konvergiert. Nehmen wir nun an, dass es ein solches $\nu_1 \leqslant n$ gibt, dass S_{ν_1-1} noch auftritt, aber S_{ν_1} nicht mehr auftritt. Alsdann bestimmen wir als K_{ν_1-1} eine solche in einer Entfernung $\circ\!\!> 0$ von A_{ν_1-2}, B_{ν_1-2} und den Simplexen $E_1 E_2 \ldots E_{\nu_1-1}$; $E_2 E_3 \ldots E_{n+1}$; $E_1 E_3 \ldots E_{n+1}$; $\ldots$; $E_1 E_2 \ldots E_{\nu_1-2} E_{\nu_1} \ldots E_{n+1}$ gelegene „vereinigende" katalogisiertkompakte Punktkernspezies einer Gruppe von Grundsimplexen eines passenden $\mathfrak{Z}_{\alpha_{\nu_1-1}}$, dass für ein gewisses $e_{\nu_1-1} \circ\!\!> 0$ alle in einer Entfernung $\leqslant e_{\nu_1-1}$ von S_{ν_1-1} gelegenen Grundsimplexe von $\mathfrak{Z}_{\alpha_{\nu_1-1}}$ zu K_{ν_1-1} gehören, während K_{ν_1-1} zwischen seinen Durchschnitten mit den Simplexen $E_1 E_2 \ldots E_{\nu_1}$ und $E_1 E_2 \ldots E_{\nu_1-1} E_{\nu_1+1} E_{\nu_1+2} \ldots E_{n+1}$ eine Sprungweite $\circ\!\!> 0$ aufweist. Sodann wählen wir *nach dem Resultate des* § 2 als K_{ν_1-2} eine solche in einer Entfernung $\circ\!\!> 0$ von A_{ν_1-3}, B_{ν_1-3} und den Simplexen $E_1 E_2 \ldots E_{\nu_1-2}$; $E_2 E_3 \ldots E_{n+1}$; $E_1 E_3 \ldots E_{n+1}$; $\ldots$; $E_1 E_2 \ldots E_{\nu_1-3} E_{\nu_1-1} \ldots E_{n+1}$ gelegene „vereinigende"

katalogisiertkompakte Punktkernspezies einer Gruppe von Grundsimplexen eines passenden $\mathfrak{Z}_{\alpha_{\nu_1-2}}(\alpha_{\nu_1-2} \geqq \alpha_{\nu_1-1})$, dass für ein gewisses $e_{\nu_1-2} > 0$ alle in einer Entfernung $\leqq e_{\nu_1-2}$ von S_{ν_1-2} gelegenen Grundsimplexe von $\mathfrak{Z}_{\alpha_{\nu_1-2}}$ zu K_{ν_1-2} gehören, und die Durchschnitte von K_{ν_1-2} mit den Simplexen $E_1 E_2 \ldots E_{\nu_1-1}$ und $E_1 E_2 \ldots E_{\nu_1-2} E_{\nu_1} \ldots E_{n+1}$ in K_{ν_1-2} durch K_{ν_1-1} getrennt werden. Darauf wählen wir, *wiederum nach dem Resultate des* § 2, als K_{ν_1-3} eine solche in einer Entfernung > 0 von A_{ν_1-4}, B_{ν_1-4} und den Simplexen $E_1 E_2 \ldots E_{\nu_1-3}; E_2 E_3 \ldots E_{n+1}; E_1 E_3 \ldots E_{n+1}; \ldots; E_1 E_2 \ldots E_{\nu_1-4} E_{\nu_1-2} \ldots E_{n+1}$ gelegene „vereinigende" katalogisiertkompakte Punktkernspezies einer Gruppe von Grundsimplexen eines passenden $\mathfrak{Z}_{\alpha_{\nu_1-3}}(\alpha_{\nu_1-3} \geqq \alpha_{\nu_1-2})$, dass für ein gewisses $e_{\nu_1-3} > 0$ alle in einer Entfernung $\leqq e_{\nu_1-3}$ von S_{ν_1-3} gelegenen Grundsimplexe von $\mathfrak{Z}_{\alpha_{\nu_1-3}}$ zu K_{ν_1-3} gehören, und die Durchschnitte von K_{ν_1-3} mit den Simplexen $E_1 E_2 \ldots E_{\nu_1-2}$ und $E_1 E_2 \ldots E_{\nu_1-3} E_{\nu_1-1} \ldots E_{n+1}$ in K_{ν_1-3} durch K_{ν_1-2} getrennt werden. Indem wir in dieser Weise fortfahren, bestimmen wir schliesslich als K_1 eine solche in einer Entfernung > 0 von A, B, dem Punkte E_1 und dem Simplexe $E_2 E_3 \ldots E_{n+1}$ gelegene „vereinigende" katalogisiertkompakte Punktkernspezies einer Gruppe von Grundsimplexen eines passenden $\mathfrak{Z}_{\alpha_1}(\alpha_1 \geqq \alpha_2)$, dass für ein gewisses $e_1 > 0$ alle in einer Entfernung $\leqq e_1$ von S_1 gelegenen Grundsimplexe von $\mathfrak{Z}_{\alpha_1}$ zu K_1 gehören, und die Durchschnitte von K_1 mit dem Segmente $E_1 E_2$ und dem Simplexe $E_1 E_3 \ldots E_{n+1}$ in K_1 durch K_2 getrennt werden. (Hierzu ist zu bemerken, dass auch $K_{\nu_1-1}, K_{\nu_1-2}, \ldots, K_2$ „vereinigende" katalogisiertkompakte Punktkernspezies von Gruppen von Grundsimplexen von $\mathfrak{Z}_{\alpha_1}$ darstellen). Bezeichnen wir noch das Grundsimplex $E_1 E_2 \ldots E_{n+1}$ mit K, so werden auch die Durchschnitte von K mit dem Punkte E_1 und dem Simplexe $E_2 E_3 \ldots E_{n+1}$ in K durch K_1 getrennt.

Mit der obigen Konstruktion ist der Beweis von Satz 2 zurückgeführt auf den S. 150 von Bd. 142 des Journ. f. Math. formulierten Hilfssatz, dessen a.a. O. S. 151 u. 152 gegebene Herleitung auch für die intuitionistische Mathematik in Kraft bleibt[4]).

[4]) Aus den Entwickelungen des § 4 geht gleichzeitig hervor, dass der aus den Sätzen 1 und 2 zusammengesetzte "Satz von der n-dimensionalität des R_n" bestehen bleibt, wenn anstatt des schwachen der mittlere oder der starke Trennungsbegriff dem Dimensionsbegriffe zu Grunde gelegt wird.

Mathematics. — *"On transformations of projective spaces"*. By Prof. L. E. J. BROUWER.

(Communicated at the meeting of May 29, 1926).

In my article: *"Ueber Abbildung von Mannigfaltigkeiten"* (Math. Ann. 71, p. 97—115) I have proved the following theorem:

A uniform continuous transformation without fixed point of an n-dimensional sphere into itself, has degree -1 *for even* n *and* $+1$ *for odd* n.

Since it appears that the specialisation of this theorem which gives an analogous property of uniform continuous transformations of n-dimensional projective spaces into themselves, and of which a particular case had already been published in these Proceedings XI, p. 798, is not to that degree, as I then thought permissible to assume, self evident to all readers, it shall be set forth here in full detail.

In the first place let there be given a uniform continuous transformation τ of a $(2n-1)$-dimensional projective space E into itself. We provide E with a positive indicatrix and with an elliptic metric. Let S be the $(2n-1)$-dimensional sphere provided with a positive indicatrix and a metric, obtained by duplication of E. Let P be a point of E, P_1 and P_2 the corresponding points of S, P' the image of P in E under τ, and P_1' and P_2' the points of S corresponding to P'. Let τ_1 be the uniform continuous transformation of S into itself which brings P_1 into P_1', and which becomes τ by the folding of S into E[1]). Then the volume of the image of S under τ_1 (measured by the volume of the image of S under the simplicial approximations of τ_1) is twice as much as the volume of the image of E under τ. Since, however, the volume of S is also twice that of E, the degree of τ_1 appears to be equal to that of τ. Further, since the absence of a fixed point for τ implies the absence of one for τ_1, *the transformation* τ, *if exhibiting no fixed point must be necessarily of degree* $+1$.

On the other hand there exist arbitrarily small *congruent* transformations of S into itself without fixed point. To these correspond arbitrarily

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[1]) The existence of τ_1 is due to the fact that to a circuit of S passing through P_1 corresponds a contractible circuit of E passing through P the image of which under τ is a contractible circuit of E passing through P' which corresponds to a circuit of S passing through P_1'.

The antipodal point-pairs of S become under τ_1 again antipodal point-pairs or simple points, according as uncontractible circuits of E become under τ again uncontractible or contractible.

small congruent transformations of E into itself without fixed point. *Thus there exist uniform continuous transformations of E into itself of degree $+1$ and without fixed point.*

In the second place let there be given a uniform continuous transformation τ of a $2n$-dimensional projective space E into itself. We provide E with an elliptic metric. Let S be the $2n$-dimensional sphere provided with a metric, obtained by duplication of E. We provide S with a positive indicatrix. Let P be a point of E, P_1 and P_2 the corresponding points of S, P' the image of P in E under τ, and P_1' and P_2' the points of S corresponding to P'. Let τ_1 and τ_2 be the uniform continuous transformations of S into itself which carry P_1 into P_1' and P_2' respectively, and which become τ by the folding of S into E. Then corresponding image simplexes of corresponding simplicial approximations of τ_1 and τ_2 have equal volumes of opposite signs; thus τ_1 and τ_2 have equal degrees of opposite signs, and thus either τ_1 or τ_2 has a fixed point. *Then, however, τ also must have a fixed point* [2]).

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[2]) Dr. HOPF points out to me that a uniform continuous transformation of a $(2n-1)$-dimensional projective space E into itself possesses at least two invariant points, if its degree is odd and $\neq +1$. We can add that on the other hand a uniform continuous transformation of a $2n$-dimensional projective space E into itself has at least two invariant points, if its *absolute degree* (i.e. the absolute value of the degrees of the correspondent transformations of the sphere S duplicating E) is ≥ 2.

Mathematics. — *"Die intuitionistische Form des* HEINE-BORELschen *Theorems"*. By Prof. L. E. J. BROUWER.

(Communicated at the meeting of June 26, 1926).

In der klassischen Mathematik gilt für einen kompakten metrischen Raum R folgende Eigenschaft:

HEINE-BORELSCHES THEOREM: *Wenn in R jedem Punkte einer kompakten Teilspezies R' von R eine Umgebung dieses Punktes zugeordnet ist, so besteht eine endliche Anzahl dieser Umgebungen, in denen R' vollständig enthalten ist.*

Bei einer früheren Gelegenheit [1]) habe ich gezeigt, dass diese Eigenschaft in ihrer allgemeinen Form in der intuitionistischen Mathematik unrichtig ist. Hier werde ich eine Spezialisierung des Theorems herleiten, welche auch in der intuitionistischen Mathematik in Kraft bleibt.

Es sei S eine katalogisiertkompakte Spezies [2]), P ein Element von S, n eine natürliche Zahl. Eine Teilspezies von S, zu der jedes in einer Entfernung $\leqslant 4^{-n}$ von P gelegene Element von S gehört, nennen wir eine *n-Umgebung von P in S*. Eine Teilspezies von S, welche für eine passende natürliche Zahl n eine n-Umgebung von P ist, nennen wir eine *Umgebung von P in S*.

Es sei S' eine in bezug auf S katalogisierte kompakte Teilspezies von S, und es sei mittels eines Gesetzes w jedem Element P' von S' eine Umgebung $\omega(P')$ von P' in S zugeordnet. In der in diesen Proceedings XXIX, S. 856 angegebenen Weise bestimmen wir zu S' für jede natürliche Zahl n die α'_n-Elemente, und sodann die mit S' „zusammenfallende" finite Menge M'. Alsdann ist jedem Elemente e' von M' infolge w eine Umgebung in S eines mit e' zusammenfallenden Elementes P' von S', und somit gleichzeitig eine natürliche Zahl $m(e')$, mittels deren die betreffende Umgebung als $m(e')$-Umgebung charakterisiert wird, zugeordnet. Auf Grund der in diesen Proceedings XXVII, S. 192 (vgl. auch S. 646) hergeleiteten Eigenschaft besteht dann, bei Variierung von e' innerhalb M', für $m(e')$ nur eine endliche Anzahl von Werten, und darunter ein Maximalwert m_1-1. Zum Gesetze w gehört also eine solche natürliche Zahl m_1, dass für jedes Element P' von S' die Umgebung $\omega(P')$ eine (m_1-1)-Umgebung ist.

Wir bezeichnen die Anzahl der α'_{m_1}-Elemente mit d_{m_1} und die α'_{m_1}-Elemente selber der Reihe nach mit ${}_\nu\alpha'_{m_1}$ $(1 \leqslant \nu \leqslant d_{m_1})$. Für jedes ν wählen

[1]) Vgl. Wis- en Natuurkundig Tijdschrift 2, S. 4; Journ. f. Math. 154, S. 4—5.

[2]) Vgl. diese Proceedings XXIX, S. 856.

wir als $P({}_{\nu}a'_{m_1})$ ein in einer Entfernung $\leqq c <_{\circ} \frac{3}{2} \cdot 4^{-m_1}$ von ${}_{\nu}a'_{m_1}$ gelegenes Element von S'. Alsdann liegt ein beliebiges Element von S', weil es in einer Entfernung $<_{\circ} 4^{-m_1}$ von einem der ${}_{\nu}a'_{m_1}$ gelegen ist, in einer Entfernung $<_{\circ} \frac{5}{2} \cdot 4^{-m_1}$ von einem der $P({}_{\nu}a'_{m_1})$, liegt also innerhalb eines der $\omega\{P({}_{\nu}a'_{m_1})\}$.

Hiermit ist folgende Eigenschaft bewiesen:

INTUITIONISTISCHES ÜBERDECKUNGSTHEOREM: *Wenn in der katalogisiertkompakten Spezies S durch ein Gesetz w einem beliebigen Elemente e einer in bezug auf S katalogisierten kompakten Teilspezies S' von S eine Umgebung von e in S zugeordnet ist, so kann eine endliche Anzahl dieser Umgebungen angegeben werden, in denen S' vollständig enthalten ist*[3]).

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[3]) Die Aussage des Satzes bleibt bestehen, wenn durch w nicht einem beliebigen Elemente e von S', sondern nur einem beliebigen (eine positiv-konvergente unbegrenzte Folge von verschiedenen Elementen einer katalogisierten Folge darstellenden) Elemente eines beliebigen Elementes e von S' eine Umgebung von e in S zugeordnet ist (ohne dass dabei notwendigerweise den verschiedenen Elementen von e die gleiche Umgebung zugeordnet zu sein braucht).